POISSON ALLOCATIONS WITH BOUNDED CONNECTED CELLS

ALEXANDER E. HOLROYD AND JAMES B. MARTIN

ABSTRACT. Given a homogenous Poisson point process in the plane, we prove that it is possible to partition the plane into bounded connected cells of equal volume, in a translation-invariant way, with each point of the process contained in exactly one cell. Moreover, the diameter D of the cell containing the origin satisfies the essentially optimal tail bound $\mathbb{P}(D > r) < c/r$. We give two variants of the construction. The first has the curious property that any two cells are at positive distance from each other. In the second, any bounded region of the plane intersects only finitely many cells almost surely.

1. INTRODUCTION

Let Π be a simple point process on $\mathbb{R}^d$. Its **support** is the random set of points $[\Pi] := \{x \in \mathbb{R}^d : \Pi(\{x\}) = 1\}$. Let $\mathcal{L}$ denote Lebesgue measure or volume on $\mathbb{R}^d$. An **allocation** of Π (to $\mathbb{R}^2$) is a random measurable map $\Phi : \mathbb{R}^d \to \mathbb{R}^d \cup \{\infty\}$ such that almost surely $\Phi(x) \in [\Pi]$ for $\mathcal{L}$-almost every $x \in \mathbb{R}^d$, and $\Phi(x) = x$ for all $x \in [\Pi]$. For a point $x \in [\Pi]$, the set $\Phi^{-1}(x)$ is called the **cell** of x. (The reason for allowing a null set to be mapped to ∞ is to avoid uninteresting complications concerning boundaries of cells.) An allocation Φ is **translation-invariant** if for every $y \in \mathbb{R}^d$, the map $x \mapsto \Phi(x - y) + y$ has the same law as Φ.

Of particular interest are translation-invariant **fair** allocations, in which all cells have equal volume. Such allocations were introduced in [7] as a tool in the construction of shift-couplings of Palm processes. Several specific choices of allocation have been studied in depth [1, 2, 3, 4, 5, 8, 9, 10]. A particular focus is on bounding the diameter of a typical cell, for allocations to a homogenous Poisson point process.

In the plane $\mathbb{R}^2$, it is natural to ask whether all cells can be connected sets. (This is clearly impossible in $\mathbb{R}$, while in $\mathbb{R}^d$ for $d \geq 3$ it is straightforward to modify any allocation to make the cells connected).

Krikun [8] constructed the first translation-invariant fair allocation of a Poisson process to $\mathbb{R}^2$ with connected cells (answering a question in [7]), but was unable to determine whether its cells are bounded. Here we construct an allocation whose cells are both connected and bounded, answering a question posed by Scott Sheffield and Yuval Peres (personal communications).

Theorem 1. *Let Π be a homogeneous Poisson point process of intensity 1 on $\mathbb{R}^2$. There exists a translation-invariant allocation of Π in which almost surely each cell is a bounded, connected set of area 1 that contains the allocated point. Moreover, the diameter D of the cell containing the origin satisfies $\mathbb{P}(D > t) < c/t$ for some c and all $t > 0$, and in addition we may choose either one of the following properties:*

    (a) *any two cells are at non-zero distance from each other; or*
    (b) *any bounded set in $\mathbb{R}^2$ intersects only finitely many cells.*

It is easily seen that no allocation can satisfy both (a) and (b): (a) implies that the line segment joining any two points of $[\Pi]$ intersects infinitely many cells, in contradiction to (b). In the above, the **diameter** of a set $A \subseteq \mathbb{R}^2$ is $\sup_{x,y \in A} \|x - y\|$, where $\|\cdot\|$ denotes the Euclidean norm. The power -1 of t in the tail bound cannot be improved: any translation-invariant fair allocation of a homogenous Poisson process satisfies $\mathbb{E}D = \infty$; see [7].

In contrast with the allocations considered in [1, 3, 8], those that we provide are not especially canonical. Rather, the point is that, armed with appropriate tools, it is not difficult to construct allocations with a variety of desirable properties. The two parts (a) and (b) will use similar constructions, with the first being slightly simpler. Our allocations are not deterministic functions of the point process Π, but require additional randomness. See e.g. [5] for more on this distinction (especially in the context of matchings). It remains an open question to prove the existence of a translation-invariant fair allocation with bounded connected cells in $\mathbb{R}^2$ that is a deterministic function of the Poisson process. It is plausible this could be done by combining our methods with deterministic hierarchical partitioning techniques as in e.g. [6, 11, 12].

2. Rational polyominos

We will construct the cells of the allocations iteratively. To do so, we want the previously constructed cells to be well-behaved subsets of the plane, while still allowing sufficient flexibility in the construction of new cells. The following definition strikes the appropriate balance.

A **rational polyomino** is a union of finitely many closed rational rectangles of the form

$$[a, b] \times [c, d] \subset \mathbb{R}^2, \quad a, b, c, d \in \mathbb{Q}.$$

By taking the least common denominator, a rational polyomino can also be expressed as a union of squares

$$(1) \qquad \frac{1}{m} \bigcup_{z \in S} \big(z + [0, 1]^2 \big),$$

for some positive integer m and some finite $S \subseteq \mathbb{Z}^2$. We write A^o for the topological interior of a set $A \subseteq \mathbb{R}^2$, and $\overline{A}$ for the closure. We call a rational polyomino **simple** if its interior and its complement are both connected, or equivalently if both the set S and its complement $\mathbb{Z}^2 \setminus S$ induce connected subgraphs of the nearest-neighbour lattice $\mathbb{Z}^2$ (the graph in which vertices $x, y \in \mathbb{Z}^2$ are joined by an edge whenever $\|x - y\|_1 = 1$).

The next lemma says that we can find a simple rational polynomial of any suitable area that contains one given set but avoids others. See Figure 1 for an illustration.

Lemma 2. *Let A be a simple rational polyomino, and let B and $D_1, \ldots, D_r$ be pairwise disjoint subsets of A°, each of which is either a simple rational polyomino or a singleton. Then, for any rational ρ with $\mathcal{L}B < \rho < \mathcal{L}(A \setminus \bigcup_i D_i)$, there exists a simple rational polyomino C with $\mathcal{L}C = \rho$ and*

$$B \subset C \subset A^\circ \setminus \bigcup_i D_i.$$

Proof. We first observe that any singletons among the given sets may be replaced with simple rational polyominos. Let k be a positive integer, and, for each singleton set $D_i = \{x_i\}$, let D_i' be the union of all squares of the form $k^{-1}([0, 1]^2 + z)$ for $z \in \mathbb{Z}^2$ that contain the point x_i (at most 4 of them). For non-singleton sets D_j let $D_j' = D_j$. Similarly define B' in terms of B. For k sufficiently large, $D_1', \ldots, D_r'$ and B' are pairwise disjoint subsets of A°, and $\mathcal{L}B' < \rho < \mathcal{L}(A \setminus \bigcup_i D_i')$. Therefore, it suffices to prove the lemma in the case when there are no singletons.

There exists an integer m such that each of the polyominos A, B, and $D_1, \ldots, D_r$ can be expressed as a union of squares of side $1/m$ as in (1). Thus, let $K, L \subset \mathbb{Z}^2$ be such that

$$(2) \quad \overline{A \setminus \bigcup_i D_i \setminus B} = \frac{1}{m} \bigcup_{z \in K} \big(z + [0, 1]^2 \big); \qquad B = \frac{1}{m} \bigcup_{z \in L} \big(z + [0, 1]^2 \big).$$
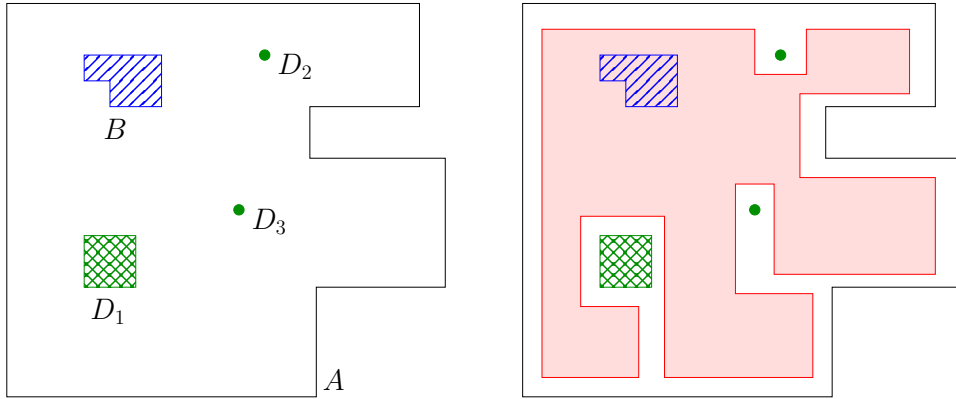
FIGURE 1. An illustration of Lemma 2. On the left, a simple rational polyomino A, containing in its interior simple rational polyominos B and D_1, and singletons D_2 and D_3. On the right, a simple rational polyomino C (shaded) within the interior of A that contains B and avoids D_1, D_2, D_3.

Note that K and L are disjoint. Both L and its complement are connected (as subsets of $\mathbb{Z}^2$), while K is connected but its complement need not be.

We now further subdivide the squares in (2). Given rational $s, t \in (0, 1)$ and $z \in \mathbb{Z}^2$, consider the rectangle of area st within $z + [0, 1]^2$ given by

$$Q_z^{s,t} := z + \left[\tfrac{1}{2} - \tfrac{s}{2}, \tfrac{1}{2} + \tfrac{s}{2}\right] \times \left[\tfrac{1}{2} - \tfrac{t}{2}, \tfrac{1}{2} + \tfrac{t}{2}\right].$$

Let w be an element of L that is adjacent in $\mathbb{Z}^2$ to some element of K. This is possible because $K \cup L$ corresponds to $\overline{A \setminus \bigcup_i D_i}$ and is therefore connected. Now take a spanning tree of the set $K \cup \{w\}$ in $\mathbb{Z}^2$. Consider the set that comprises the rectangle $Q_u^{s,t}$ for each $u \in K$, together with the rectangle that is the convex hull of $Q_u^{s,t} \cup Q_v^{s,t}$ for each edge (u, v) of the spanning tree. Take the union of this set with B, and call it $C^{s,t}$.

The set $m^{-1}C^{s,t}$ is a simple rational polyomino that contains B and is contained in $A^\circ \setminus \bigcup D_i$. To complete the proof, we will show that we can choose rational $s, t \in (0, 1)$ so that $\mathcal{L}(m^{-1}C^{s,t}) = \rho$, which is to say $\mathcal{L}C^{s,t} = m^2\rho$.

Note that $\mathcal{L}C^{s,t}$ can be expressed as the sum of the following terms: $\mathcal{L}B$, plus st for each element of K, plus $s(1-t)/2$ for each horizontal edge of the tree that is incident to w, and $s(1 - t)$ for each other horizontal edge of the tree, plus similarly $(1-s)t/2$ or $(1-s)t$ for each

vertical edge. Therefore,

$$(3) \qquad \mathcal{L}C^{s,t} = \alpha st + \beta s + \gamma t + \delta$$

for some rational $\alpha, \beta, \gamma, \delta$ that do not depend on s, t. Moreover, since the number of edges of the tree equals the number of elements of K, and at least one edge is incident to w, we have $\alpha, \delta > 0$ and $\beta, \gamma \geq 0$.

The expression in (3) is continuous and strictly increasing in both s and t. As $(s,t) \to (0,0)$ we have $m^{-2}\mathcal{L}C^{s,t} \to \mathcal{L}B < \rho$, while as $(s,t) \to (1,1)$ we have $m^{-2}\mathcal{L}C^{s,t} \to \mathcal{L}(A \setminus \bigcup D_i) > \rho$. Hence, writing $s_0 = \sup\{s : m^{-2}\mathcal{L}C^{s,0} < \rho\}$ and $s_1 = \inf\{s : m^{-2}\mathcal{L}C^{s,1} > \rho\}$, we have $s_0 < s_1$. Fix a rational $s \in (s_0, s_1)$; we have $s \in (0,1)$ and $m^{-2}\mathcal{L}C^{s,0} < \rho < m^{-2}\mathcal{L}C^{s,1}$. Thus there exists $t \in (0,1)$ with $m^{-2}\mathcal{L}C^{s,t} = \rho$; by (3), this t must be rational. $\qquad\square$

3. Non-touching allocation

Proof of Theorem 1(a). We first construct an allocation that is invariant under all translations by elements of $\mathbb{Z}^2$ and whose cells have the claimed properties; we will obtain a fully translation-invariant version by translating both the allocation and the point process by a uniformly random element of $[0,1)^2$.

The cells of our allocation will be simple rational polyominos. We first define a sequence of successively coarser partitions of $\mathbb{R}^2$ into squares in a $\mathbb{Z}^2$-invariant way. (This construction is standard; see e.g. [5]). Let $(\alpha_i)_{i=0,1,\dots}$ be i.i.d. uniformly random elements of the discrete cube $\{0,1\}^2$, independent of Π. Given the sequence (α_i), define a k-**block** for $k \geq 0$ to be any set of the form $[0,2^k)^2 + z2^k + \sum_{i=0}^{k-1}\alpha_i 2^i$, for $z \in \mathbb{Z}^2$. (So a $(k+1)$-block is the disjoint union of four k-blocks, and every k-block has area 4^k.)

We now construct an allocation in a sequence of steps $k = 1, 2, \dots$. At step k we will construct some cells, each of which will be confined within the interior of some k-block. For step 1 we proceed as follows. For each 1-block R, let $x_1, x_2, \dots, x_s$ be the points of $[\Pi] \cap R$, enumerated lexicographically, say. If $s \geq 1$, let C_1 be a rational polyomino of area 1 that satisfies $x_1 \in C_1 \subset R^\circ$ and that avoids the other points $x_2, \dots, x_s$; this exists by Lemma 2, with $A = \overline{R}$. Declare C_1 be the cell allocated to the point x_1. Now if $s \geq 2$, similarly find a rational polyomino C_2 of area 1 in R that contains x_2 and avoids C_1 and $x_3, \dots, x_s$, and allocate it to x_2. Similarly if $s \geq 3$, allocate to x_3 a cell avoiding $C_1 \cup C_2$ and $x_4, \dots, x_s$. In each case, this is possible by Lemma 2, because the total area required for C_1, C_2, C_3 is 3, which is strictly less than $\mathcal{L}R = 4$.

For step k we proceed as follows. Let R be a k-block, and enumerate the unallocated points of $[\Pi] \cap R$ lexicographically. For each in turn, use Lemma 2 to choose a rational polyomino of area 1 in R° that contains the point, and avoids all other points of $[\Pi] \cap R$ and all previously chosen cells that intersect R (all such cells are in fact subsets of R). Continue until either we run out of unallocated points in R, or the total area of all the cells in R reaches $\mathcal{L}R - 1$. Do this for each k-block.

After all steps have been completed as above, define an allocation Ψ by setting $\Psi(y) = x$ if y is in the cell assigned to $x \in [\Pi]$, and $\Psi(y) = \infty$ for all other $y \in \mathbb{R}^2$. It is clear that each cell of Ψ is either empty or a simple rational polyomino of area 1 that contains the corresponding point of Π. It is also clear that Ψ has the required $\mathbb{Z}^2$-invariance property provided the cells are chosen according to fixed translation-invariant rules; this is possible since all the steps in the proof of Lemma 2 can be carried out in a translation-invariant way. Every cell is a closed set, and hence any two non-empty cells are at positive distance from each other, since they do not intersect.

Now let U be a uniformly random element of the unit square $[0,1)^2$, independent of (Π, Ψ), and define a translated allocation Ψ' by $\Psi(x) := U + \Psi(x - U)$. Then Ψ' is a fully translation-invariant allocation of the translated point process Π' defined by $\Pi'(A) := \Pi(A + U)$ (which is a Poisson process). It remains to show that every point of the process is allocated a non-empty cell, and that almost every $x \in \mathbb{R}^2$ is allocated to some cell, and that the claimed diameter bound holds.

Let D be the diameter of the cell of Ψ' containing the origin 0, if it exists, and let $D = \infty$ if $\Psi'(0) = \infty$. Then D has the same law as the diameter of the cell of Ψ containing a uniformly random point U in $[0,1)^2$. Note that any cell that is constructed at step k or earlier lies entirely within some k-block, and therefore has diameter at most $2^k \sqrt{2}$. By $\mathbb{Z}^2$-invariance, the probability that U is allocated by step k equals the expected proportion of the k-block containing $[0,1)^2$ that is allocated by step k. Since the positions of blocks are independent of Π, this expected proportion remains the same if we condition the k-block to have a specific position, say $S := [0, 2^k)^2$. The total area allocated within S by step k is precisely $\min\{\Pi(S), 4^k - 1\}$ (since new cells are added while there are unallocated points until their total area is one less than the area 4^k of S). Thus for all integers $k \geq 1$,

$$\mathbb{P}\big(D > 2^k \sqrt{2}\big) \leq 1 - 4^{-k}\, \mathbb{E} \min\big\{\Pi(S), 4^k - 1\big\}$$
$$\leq 4^{-k}\left(1 + \mathbb{E}\big[(4^k - \Pi(S))^+\big]\right)$$

Since $\Pi(S)$ is Poisson distributed with mean 4^k, we have $\mathbb{E}[(4^k - \Pi(S))^+] \leq C\sqrt{4^k}$ for some C, and it follows that $\mathbb{P}(D > t) < c/t$ as claimed.

In particular the above implies that $D < \infty$ almost surely, and so almost every $x \in \mathbb{R}^2$ is assigned to some cell by Ψ'. Since each cell has area 1, a standard mass-transport argument (see e.g. [3, 5]) then implies that the process of those points of Π' that are allocated cells has intensity 1. Since Π' has intensity 1, this shows that almost surely every point of Π' is allocated. $\qquad\square$

4. Locally finite allocation

Proof of Theorem 1(b). As at the beginning of the proof of part (a), we define a hierarchy of k-blocks using an i.i.d. sequence (α_i). As in the previous proof, it suffices to construct an appropriate allocation that is invariant under $\mathbb{Z}^2$, and then apply a random translation.

For each block we define an inner block. Let $(\eta_k)_{k\geq 1}$ be a strictly decreasing sequence of rational numbers in $(\frac{1}{2}, 1)$ with $\eta_k \downarrow \frac{1}{2}$ as $k \to \infty$. If B_k is a k-block then $B_k = (a, b) + [0, 2^k)^2$ for some point $(a, b) \in \mathbb{Z}^2$. Define its **inner block** I_k by $I_k = (a, b) + [\eta_k, 2^k - \eta_k]^2$. Thus I_k is a square of side $2^k - 2\eta_k$ with the same centre as B_k. Define also $M_k = B_k \cap I_{k+1}$, where I_{k+1} is the inner block of the $(k + 1)$-block containing B_k. Thus M_k is a square of side $2^k - \eta_{k+1}$, which contains I_k in its interior (since the sequence η_k is strictly decreasing).

As in the previous proof, we construct the allocation in a sequence of steps $k = 1, 2, \ldots$. At step k we add some cells to the allocation, with each such cell confined to the interior of some k-block.

At step k we treat each k-block separately. Let B_k be a k-block. The following statement plays the role of induction hypothesis: at the beginning of step k, the closure of the union of the previously allocated cells in B_k is a union of disjoint simple rational polyominos contained in the interior of I_k. In particular, the complement with respect to I_k of this set is connected.

The allocations during step k will be carried out in such a way that at the end of step k, the following holds: the closure of the union of the allocated cells in B_k forms a collection of disjoint simple rational polyominos contained in M_k^o. This property implies the induction hypothesis for next level $k + 1$; for if B_{k+1} is the $(k + 1)$-block containing B_k, then its inner block I_{k+1} is made up of the square M_k together with three other analogous squares; the interiors of these squares are disjoint.

Now we explain how the allocation within B_k at step k is carried out. We say that the box B_k is **good** if

$$\tag{4} \mathcal{L}I_k < \Pi(I_k) < \mathcal{L}M_k.$$

We proceed in two different ways depending on whether or not B_k is good.

If B_k is not good, we allocate only within the inner box I_k. (By the induction hypothesis, all previously allocated cells in B_k are in the interior of I_k). In this case we proceed in the same fashion as we did in the construction of the non-touching allocation in part (a). Using Lemma 2, we add new cells in the interior of I_k one by one, each one a simple rational polyomino disjoint from previous cells. We continue until either the number of cells is $\Pi(I_k)$ or the remaining unallocated area inside I_k is at most 1.

If B_k is good, we start by finding a region lying between I_k and M_k which contains a number of points of Π exactly equal to its area. Because (4) holds, this can be done using Lemma 2, setting $\rho = \Pi(I_k)$, $B = I_k$ and $A = M_k$, and setting $D_1, \ldots, D_r$ to be the points of Π in $M_k \setminus I_k$ (and $r = 0$). In this way we find a simple rational polyomino C with $I_k \subset C \subset M_k^o$ and $\Pi(C) = \mathcal{L}(C)$.

Now we will divide up the set C to form the cells allocated to points of Π in C. Some such allocations may already have been done. All the remaining ones except the last can be done one by one, just as before, using Lemma 2. These new allocations are simple random polyominos, disjoint from previous cells, containing precisely one point of Π and contained in the interior of C; in particular the remainder of C stays connected. Finally, when one point of Π remains in C, and hence when area 1 remains to be allocated, we allocate the rest of C as the cell of the last point. The closure of this cell is a rational polyomino, and is connected but not simple.

At the end of the procedure, as in part (a) define Φ by setting $\Phi(y) = x$ whenever y is in the cell assigned to $x \in [\Pi]$, and $\Phi(y) = \infty$ otherwise. Each such cell is either empty or has area 1 and contains the corresponding point of Π. As before, by carrying out the steps of Lemma 2 in a translation-invariant way, we can ensure that Φ has the required $\mathbb{Z}^2$-invariance property.

If B_k is a good box, then the number of cells that intersect I_k is finite. Also, every point in I_k is allocated to some cell. To show that every point in $\mathbb{R}^2$ is allocated to some cell and that the allocation is locally finite as desired, it will be enough to show that with probability 1, every point is in the interior of the inner box of some good box.

Let S be any 1×1 square. The probability that B is contained in the interior of the inner box of some k-block is $(2^k - 2\eta_k - 1)^2/4^k$, which tends to 1 as $k \to \infty$. If this event holds for some k, then in fact it holds for all $k' > k$ also (since the inner box of a k-block lies within the inner box of the containing $(k+1)$-block). Hence with probability 1, this event holds for all large enough k, say $k \geq k_0$.

Now let B_k be the k-block containing S, with I_k and M_k defined as before. It will be enough to show that with probability 1, B_k is good for infinitely many k. From (2), $\mathbb{P}(B_k \text{ is good}) = \mathbb{P}(\mathcal{L}I_k < \Pi(I_k) < \mathcal{L}M_k)$. We have $\mathcal{L}I_k = (2^k - 2\eta_k)^2$, and $\mathcal{L}M_k = (2^k - \eta_{k+1})^2$. Therefore, since $\eta_k > \eta_{k+1} > \frac{1}{2}$,

$$\mathcal{L}M_k - \mathcal{L}I_k = (2^{k+1} - 2\eta_k - \eta_{k+1})(2\eta_k - \eta_{k+1}) > (2^{k+1} - 3\eta_k)\tfrac{1}{2}$$
$$> \sqrt{\mathcal{L}I_k}.$$

Since $\Pi(I_k)$ is $\text{Poisson}(\mathcal{L}I_k)$ and $\mathcal{L}I_k \to \infty$, we obtain $\mathbb{P}(B_k \text{ is good}) \geq c$ for all large enough k for some constant c (in fact, any $c < \mathbb{P}(0 < Z < 1)$ is enough, where Z is a standard Gaussian).

If the events $\{B_k \text{ is good}\}$ were independent for different k, this would be enough; however, we need to control the dependence. To do this, consider any sequence $k_1 < k_2 < \ldots$ with the following properties:

(i) $\mathbb{P}\left(\Pi\left(I_{k_n}\right) > \frac{1}{2}\sqrt{\mathcal{L}I_{k_{n+1}}} \right) < 2^{-n}$;

(ii) $\mathbb{P}\left(\mathcal{L}I_{k_{n+1}} < \Pi\left(I_{k_{n+1}} \setminus I_{k_n}\right) < \mathcal{L}I_{k_{n+1}} + \frac{1}{2}\sqrt{\mathcal{L}I_{k_{n+1}}} \right) > c'$
(where c' is a constant independent of k).

By similar arguments to the above, this is easily shown to be possible by making k_n grow quickly enough. The events in (ii) are independent for different n, so by Borel-Cantelli, with probability 1 infinitely many of them occur. The sum of the probabilites in (i) is finite, so with probability 1 only finitely many of them occur. But for any given n, if the event in (i) fails and the event in (ii) holds then $B_{k_{n+1}}$ is good. So with probability 1, there are infinitely many k for which B_k is good, as desired.

Finally, we turn to the diameter bound. At step k, the area allocated within a k-block B_k is at least $\min\left(\Pi(I_k) - 1, \mathcal{L}I_k - 2\right)$. Arguing as for the non-touching allocation in part (a), we obtain

$$\mathbb{P}(D > 2^k\sqrt{2}) \leq 1 - 4^{-k}\mathbb{E}\min\left(\Pi(I_k) - 1, \mathcal{L}I_k - 2\right).$$

where D is the diameter of the cell containing the origin, in the allocation obtained by translating Φ by a random element of $[0,1)^2$. As

before, this is easily seen to be at most $C\sqrt{4^k}$ for some $C < \infty$, giving the desired bound on the tail of D. $\qquad\square$

References

[1] S. Chatterjee, R. Peled, Y. Peres, and D. Romik. Gravitational allocation to Poisson points. *Ann. of Math. (2)*, 172(1):617–671, 2010.

[2] S. Chatterjee, R. Peled, Y. Peres, and D. Romik. Phase transitions in gravitational allocation. *Geom. Funct. Anal.*, 20(4):870–917, 2010.

[3] C. Hoffman, A. E. Holroyd, and Y. Peres. A stable marriage of Poisson and Lebesgue. *Ann. Probab.*, 34(4):1241–1272, 2006.

[4] C. Hoffman, A. E. Holroyd, and Y. Peres. Tail bounds for the stable marriage of Poisson and Lebesgue. *Canad. J. Math.*, 61(6):1279–1299, 2009.

[5] A. E. Holroyd, R. Pemantle, Y. Peres, and O. Schramm. Poisson matching. *Ann. Inst. Henri Poincaré Probab. Stat.*, 45(1):266–287, 2009.

[6] A. E. Holroyd and Y. Peres. Trees and matchings from point processes. *Electron. Comm. Probab.*, 8:17–27 (electronic), 2003.

[7] A. E. Holroyd and Y. Peres. Extra heads and invariant allocations. *Ann. Probab.*, 33(1):31–52, 2005.

[8] M. Krikun. Connected allocation to Poisson points in $\mathbb{R}^2$. *Electron. Comm. Probab.*, 12:140–145, 2007.

[9] R. Markó and A. Timár. A poisson allocation of optimal tail. 2011. arXiv:1103.5259.

[10] F. Nazarov, M. Sodin, and A. Volberg. Transportation to random zeroes by the gradient flow. *Geom. Funct. Anal.*, 17(3):887–935, 2007.

[11] T. Soo. Translation-equivariant matchings of coin flips on $\mathbb{Z}^d$. *Adv. in Appl. Probab.*, 42(1):69–82, 2010.

[12] A. Timar. Invariant matchings of exponential tail on coin flips in $\mathbb{Z}^d$. 2009. arXiv:0909.1090.

Alexander E. Holroyd, Microsoft Research, 1 Microsoft Way, Redmond, WA 98052, USA

E-mail address: `holroyd at microsoft.com`

James B. Martin, Department of Statistics, 1 South Parks Rd, Oxford OX1 3TG, UK

E-mail address: `martin at stats.ox.ac.uk`