Existence of a Phase Transition for Entanglement Percolation

Alexander E. Holroyd

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Abstract

We consider the bond percolation model on the three-dimensional cubic lattice, in which individual edges are retained independently with probability p. We shall describe a subgraph of the lattice as 'entangled' if, roughly speaking, it cannot be 'pulled apart' in three dimensions. We shall discuss possible ways of turning this into a rigorous definition of entanglement. For a broad class of such definitions, we shall prove that for p sufficiently close to zero, the graph of retained edges has no infinite entangled subgraph almost surely, thereby establishing that there is a phase transition for entanglement at some value of p strictly between zero and unity.

1 Introduction

Consider the bond percolation model on the three-dimensional cubic lattice, in which individual edges are retained independently with probability p, and imagine the retained edges as physical connections in three dimensions made of elastic. We wish to say that a graph is 'entangled' if the corresponding elastic object cannot be 'pulled apart' into two or more parts in three-dimensional space, perhaps because it includes two linked loops of edges as in the graph illustrated in Figure 1. We shall be concerned with the question of whether or not the graph of retained edges has an infinite entangled subgraph. This question has important physical applications, in particular to models of polymers in solution, and has been studied in [6] using partly non-rigorous methods. Our aim here is to approach the problem from a mathematically rigorous standpoint.

Key words: entanglement percolation, percolation, critical points, entanglement, topological entanglement, plaquettes

AMS subject classifications: Primary 60K35; Secondary 82B43, 82B41, 05C10, 57M25

Address of author: Statistical Laboratory, University of Cambridge, 16 Mill Lane, Cambridge, CB2 1SB, United Kingdom. Email: holroyd@statslab.cam.ac.uk.

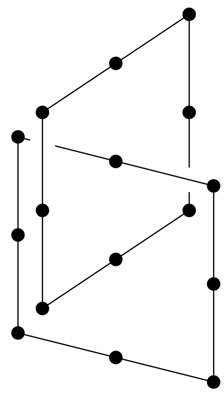

Figure 1: An example of an entangled graph.

We shall work throughout with the three-dimensional cubic lattice. As in the theory of knots (see [7]), the entanglement of arcs is intrinsically a three-dimensional affair. Our choice of the cubic lattice is largely a matter of convenience, and other choices are possible.

We shall see that there are several possible ways of formulating a mathematical definition of entanglement, and indeed, different definitions may be appropriate to different applications. Given some definition of an entangled graph, one may define the entanglement critical probability p_{e}, and it is natural to ask whether the inequalities

$$0 < p_{\mathrm{e}} < 1$$

hold. Since (according to any reasonable definition of entanglement), any connected graph is entangled, we expect the inequality

$$p_{\mathrm{e}} \leq p_{\mathrm{c}},$$

where p_{c} is the usual connectivity critical probability, and since we have $p_{\mathrm{c}} < 1$ this would establish the upper bound $p_{\mathrm{e}} < 1$. In [2], the authors describe how their general method can be used to strengthen the above inequality to $p_{\mathrm{e}} < p_{\mathrm{c}}$ (although no explicit definition of entanglement is given). In [6], the authors assert, on the basis of numerical work, that we have $p_{\mathrm{c}} - p_{\mathrm{e}} \approx 1.8 \times 10^{-7}$. If true, this would imply the inequality $p_{\mathrm{e}} > 0$, since the value of p_{c} is believed to be approximately 0.249 (see [5], page 181), and for example we have the rigorous bound $p_{\mathrm{c}} \geq 1/5$ (see [3], page 15). The aim of this paper is to give a rigorous proof of the inequality

$$p_{\mathrm{e}} > 0$$

which will apply under any reasonable definition of entanglement. (For the precise statement, see Theorem 1 in the next section). Our approach will be to prove that for p sufficiently close to zero, there is almost surely a surface homeomorphic to a sphere which encloses the origin and does not intersect any retained edges.

We shall construct such a surface using techniques related to those in [1], using the dual plaquette percolation model.

We remark that it may be tempting to conclude that the inequality $p_{\mathrm{e}} > 0$ is 'obvious'. However, it should be noted that certain similar inequalities arising in bootstrap percolation turn out to be false. For more details see [9].

The forthcoming publication [4] contains a detailed treatment of the various possible definitions of entanglement, together with some further rigorous results on entanglement percolation.

2 Statement of problem and results

We define $\mathbb{Z}^3$ to be the set of all 3-vectors of integers $x = (x_1, x_2, x_3)$, and define the **cubic lattice** to be

$$\mathbb{L} = \{\{x, y\} \subseteq \mathbb{Z}^3 : \|x - y\| = 1\},$$

where $\|\cdot\|$ denotes Euclidean distance. We refer to the members of $\mathbb{Z}^3$ as **vertices** and the members of $\mathbb{L}$ as **edges**. The **origin** is the vertex $O = (0, 0, 0)$. By a **graph** we mean a non-empty set of edges $G \subseteq \mathbb{L}$, and by a **subgraph** we mean a non-empty subset of a graph. We say that a graph G **contains** a vertex x if there exists $e \in G$ with $x \in e$. We wish to consider a random subgraph K of $\mathbb{L}$ in which each edge of $\mathbb{L}$ is included with probability p, and distinct edges are treated independently. To be precise, we define the sample space $\Omega_{\mathbb{L}} = \{0, 1\}^{\mathbb{L}}$, equipped with the product σ-field. For $p \in [0, 1]$ we define $P_p^{\mathbb{L}}$ to be the product measure on $\Omega_{\mathbb{L}}$ with parameter p. We define the random variable K by $K(\omega) = \{e \in \mathbb{L} : \omega(e) = 1\}$. We shall refer to this set-up as the **bond percolation model with parameter** p.

In the standard theory of connectivity percolation, one is concerned with the connected components of the graph K. In particular, we define the **connectivity critical probability**

$$p_{\mathrm{c}} = \sup\{p : P_p^{\mathbb{L}}(K \text{ has an infinite connected subgraph containing } O) = 0\}.$$

It is straightforward to show that for $p < p_{\mathrm{c}}$ we have

$$P_p^{\mathbb{L}}(K \text{ has an infinite connected subgraph}) = 0,$$

while for $p > p_{\mathrm{c}}$ we have

$$P_p^{\mathbb{L}}(K \text{ has an infinite connected subgraph}) = 1.$$

It can be shown that we have

$$0 < p_{\mathrm{c}} < 1,$$

so that there is a genuine 'phase transition' between these two regimes. See [3] (for example) for details.

Before giving any rigorous results about entanglement it is necessary to formulate a precise definition. As mentioned in the introduction, the idea which we wish to formalise is that a graph should be considered entangled if it cannot be 'pulled apart'. However, we shall see that there are potentially several different ways of formalising this notion, which give rise to different definitions of entanglement. We shall explain that our main result applies whichever definition we choose.

We start with some notation. For a finite set $A = \{a_1, \dots, a_k\} \subseteq \mathbb{R}^3$, we define the **(closed) convex hull**

$$\langle A \rangle = \left\{ \sum_i \lambda_i a_i : 0 \leq \lambda_i \leq 1 \text{ for each } i, \text{ and } \sum_i \lambda_i = 1 \right\}.$$

Note that if $e \in \mathbb{L}$ is an edge, then $\langle e \rangle$ is the closed unit line segment which we normally associate with e. For a graph $E \subseteq \mathbb{L}$ we write $[E] = \bigcup_{e \in E} \langle e \rangle$. For any closed set $R \subseteq \mathbb{R}^3$ and any $\epsilon > 0$ we define the (open) **ϵ-neighbourhood** of R, which we denote by $R^{\{\epsilon\}}$, to be the set of points at Euclidean distance strictly less than ϵ from R:

$$R^{\{\epsilon\}} = \{x \in \mathbb{R}^3 : \|x - r\| < \epsilon \text{ for some } r \in R\}.$$

We shall make use of various topological objects, which we define here. By a **d-ball** or **d-sphere** we mean a closed d-dimensional simplicial complex in $\mathbb{R}^3$ which is homeomorphic to $\{x \in \mathbb{R}^d : \|x\| \leq 1\}$ or $\{x \in \mathbb{R}^{d+1} : \|x\| = 1\}$ respectively. (Loosely speaking, a simplicial complex is a compact set which is the union of finitely many polyhedral pieces. See [8] for a definition.) We use the synonyms **point** for 0-ball, **arc** for 1-ball, **disc** for 2-ball, **ball** for 3-ball, **loop** for 1-sphere and **sphere** for 2-sphere. Let R be a d-ball and let ϕ be a homeomorphism from $\{x \in \mathbb{R}^d : \|x\| \leq 1\}$ to R. We define the **boundary** of R, written ∂R, to be the $(d-1)$-sphere $\phi(\{x \in \mathbb{R}^d : \|x\| = 1\})$. The **interior** of R is the set $R \setminus \partial R$. If S is a sphere, we define its **inside** to be the bounded connected component of $\mathbb{R}^3 \setminus S$, and its **outside** to be the unbounded connected component of $\mathbb{R}^3 \setminus S$. If S is a sphere and R is a subset of $\mathbb{R}^3$, we say that S **separates** R if R intersects both the inside and the outside of S but does not intersect S.

We have already remarked that a subgraph of $\mathbb{L}$ consisting of two linked loops should be regarded as entangled, and we can also give examples of infinite subgraphs of $\mathbb{L}$, such as that illustrated in Figure 2, which should clearly be regarded as entangled. However, for general infinite graphs the situation is not so clear. To illustrate this, note that we must for example decide whether or not we wish to regard each of the infinite graphs illustrated in Figure 3 as entangled. Our approach will be based on the following observation. If $E \subseteq \mathbb{L}$ is any graph

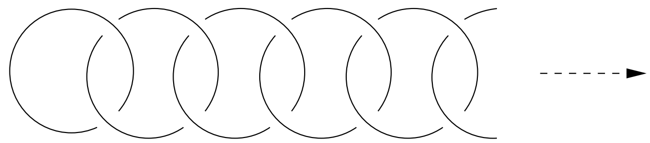

Figure 2: An infinite entangled graph.

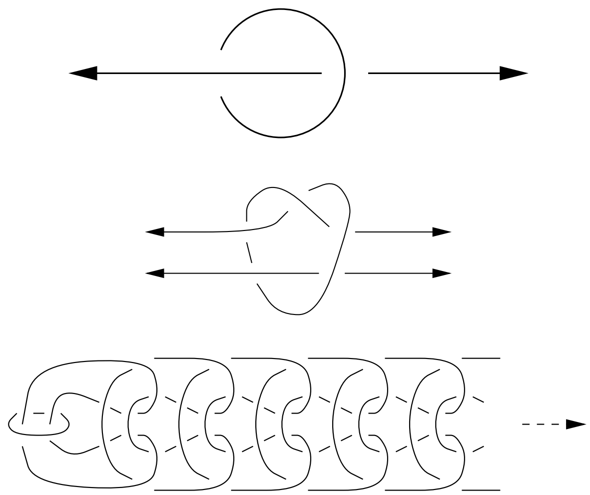

Figure 3: Are these graphs entangled?

which is entangled according to some 'reasonable' definition, then there exists no sphere which separates $[E]$. This idea corresponds with the intuitive notion of entanglement in the following way. If $[E]$ were separated by a sphere S, we could 'shrink' S and its contents, and then separate the graph into two parts by moving the sphere a large distance away. Note for example that none of the graphs illustrated in Figure 3 is separated by a sphere. It is important that we require S to be a *sphere*, and not just any closed surface. To see this, note that if E is the graph in Figure 1, then there is a torus which 'separates' the two loops.

In order to formalise the above remarks, consider any set $\mathcal{E}$ of graphs in $\mathbb{L}$. (We think of $\mathcal{E}$ as a candidate for the set of all entangled sets according to some definition). We say that $\mathcal{E}$ is an **e-system** if the following properties hold.

(i) $\mathcal{E}$ contains every connected subgraph of $\mathbb{L}$.

(ii) $\mathcal{E}$ is translation-invariant; that is, for every $A \in \mathcal{E}$ and $x \in \mathbb{Z}^3$ we have $A + x \in \mathcal{E}$.

(iii) If $A \in \mathcal{E}$ then no sphere separates $[A]$.

(iv) The event $\{K$ has an infinite subgraph lying in $\mathcal{E}$ and containing $O\}$ is measurable; that is, it lies in the usual product σ-field.

We remark that condition (iv) in the above definition is indeed needed; one may construct sets $\mathcal{E}$ which satisfy (i), (ii) and (iii) but not (iv). We claim that under any reasonable definition of entanglement, the set of all entangled graphs is an e-system. We stress that there are several families of graphs which fall into this category (one simple example is the set of all graphs which are not separated by any sphere). A detailed treatment of the possible definitions of entanglement (which are related to the issue of 'boundary conditions') appears in [4]. The results presented in this paper apply to *any* e-system.

For an e-system $\mathcal{E}$ we define the associated critical probability

$$p_{\mathrm{e}}(\mathcal{E}) = \sup\{p : P_p^{\mathbb{L}}(K \text{ has an infinite subgraph lying in } \mathcal{E} \\ \text{and containing } O) = 0\}.$$

It is straightforward to check that the following statements hold. Using property (ii), for $p < p_{\mathrm{e}}(\mathcal{E})$ we have

$$P_p^{\mathbb{L}}(K \text{ has an infinite subgraph lying in } \mathcal{E}) = 0,$$

while for $p > p_{\mathrm{e}}(\mathcal{E})$ we have

$$P_p^{\mathbb{L}}(K \text{ has an infinite subgraph lying in } \mathcal{E}) = 1.$$

By property (i) we have $p_{\mathrm{e}}(\mathcal{E}) \leq p_{\mathrm{c}}$. (Note that we shall make no further use of properties (i) and (ii) of an e-system).

We are now ready to state our main result.

Theorem 1 *Suppose $\mathcal{E}$ is an e-system. Let $p_0 = 1/15616$. For any $p \leq p_0$ we have*

$$P_p^{\mathbb{L}}(K \textit{ has an infinite subgraph lying in } \mathcal{E} \textit{ and containing } O) = 0.$$

Hence we have

$$p_{\mathrm{e}}(\mathcal{E}) \geq p_0.$$

Together with the above observations, Theorem 1 establishes that for any e-system $\mathcal{E}$ we have the inequalities

$$0 < p_{\mathrm{e}}(\mathcal{E}) < 1,$$

so that there is a genuine phase transition between the two regimes described above.

Our approach to proving Theorem 1 will use property (iii) of an e-system; we shall prove that for p sufficiently close to zero, almost surely with respect to $P_p^{\mathbb{L}}$ there is a sphere lying in $\mathbb{R}^3 \setminus [K]$ with O in its inside. In order to do this we introduce the notion of plaquette percolation (for more details see [1], for example). We define the set

$$\mathbb{P} = \{\{a, b, c, d\} \subseteq \mathbb{Z}^3 : a, b, c, d \text{ are distinct and} \\ \|a - b\| = \|b - c\| = \|c - d\| = \|d - a\| = 1\};$$

we refer to the members of $\mathbb{P}$ as **plaquettes**. If f is a plaquette, note that the convex hull $\langle f \rangle$ is a closed square subset of $\mathbb{R}^3$ of unit side-length. For a set of plaquettes $F \subseteq \mathbb{P}$ we write $[F] = \bigcup_{f \in F} \langle f \rangle$. In the **plaquette percolation model with parameter** q we consider a random subset Q of $\mathbb{P}$ in which each plaquette is included with probability q, and distinct plaquettes are treated independently. To be precise, we define the sample space $\Omega_{\mathbb{P}} = \{0,1\}^{\mathbb{P}}$, equipped with the product σ-field. For $q \in [0,1]$ we define $P_q^{\mathbb{P}}$ to be the product measure on $\Omega_{\mathbb{P}}$ with parameter q. We define the random variable Q by $Q(\omega) = \{f \in \mathbb{P} : \omega(f) = 1\}$.

The reason for introducing plaquettes is that there is a 'duality' relationship between the plaquette and bond percolation models, which we shall describe briefly (for more details see [1]). Let $\mathbb{L}_+$ be the 'shifted' cubic lattice $\mathbb{L} + (1/2, 1/2, 1/2)$. For an edge $e \in \mathbb{L}$, let e_+ be the shifted edge $e + (1/2, 1/2, 1/2) \in \mathbb{L}_+$. Note that for each edge $e \in \mathbb{L}$, there is a unique plaquette, which we denote $f(e) \in \mathbb{P}$, such that $\langle e_+ \rangle$ intersects $\langle f(e) \rangle$; f is a bijection from $\mathbb{L}$ to $\mathbb{P}$. We make the important further observation that $f(e)$ is the unique plaquette such that $\langle e \rangle$ intersects the 1/4-neighbourhood $\langle f(e) \rangle^{\{1/4\}}$. Now suppose that $p + q = 1$, and write K_+ for the shifted random graph $K + (1/2, 1/2, 1/2) \subseteq \mathbb{L}_+$. Clearly, we may couple the measures $P_p^{\mathbb{L}}$ and $P_q^{\mathbb{P}}$ in such a way that that any edge $e \in \mathbb{L}$ is included in K if and only if the corresponding plaquette $f(e)$ is *not* included in Q. By the above remark, under this coupling, the sets $[K_+]$ and $[Q]^{\{1/4\}}$ are disjoint with probability one. Thus the plaquette model gives a useful way to construct sets which lie in $\mathbb{R}^3 \setminus [K]$.

Our main result about plaquette percolation is Theorem 2, which states that for a sufficiently high density of plaquettes, there is almost surely a sphere enclosing a specified point which is contained in the set of plaquettes Q 'thickened' by distance 1/4. This result will be proved in Section 3.

Theorem 2 *Let $p_0 = 1/15616$. For any $q \geq 1 - p_0$, in the plaquette percolation model with parameter q we have*

$$P_q^{\mathbb{P}}([Q]^{\{1/4\}} \textit{ contains a sphere with the point } (1/2, 1/2, 1/2) \\ \textit{in its inside}) = 1.$$

(Note that the event mentioned in Theorem 2 is measurable, since it may be expressed a countable union of events concerning spheres in particular finite volumes, and these are cylinder events. Analogous arguments ensure the measurability of all other similar events which we shall consider, and we shall not generally make such arguments explicit.)

We now deduce Theorem 1 from Theorem 2.

Proof of Theorem 1 Suppose $p \leq p_0$, and put $q = 1 - p$, so that $q \geq 1 - p_0$. Applying Theorem 2, almost surely with respect to $P_q^{\mathbb{P}}$, $[Q]^{\{1/4\}}$ contains a sphere, S say, with $(1/2, 1/2, 1/2)$ in its inside. Under the coupling described above, since $[Q]^{\{1/4\}}$ and $[K_+]$ are disjoint, S lies in $\mathbb{R}^3 \setminus [K_+]$. Now suppose that I were an

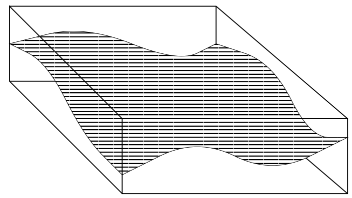

Figure 4: A disc 'spanning' a block.

infinite subgraph of K lying in $\mathcal{E}$ and containing O and let $I_+ = I+(1/2, 1/2, 1/2)$. Since I_+ is unbounded, it must contain points in the outside of S, but I_+ also contains $(1/2, 1/2, 1/2)$, which lies in the inside of S, contradicting property (iii) of an e-system. $\square$

We observe that the proof of Theorem 2 which we shall give may be adapted to give a quantitative probabilistic upper bound on size of the sphere enclosing the origin, and hence on the size of the entangled component at the origin. Specifically, define the **radius** of a set $R \subseteq \mathbb{R}^3$ containing the origin to be $\sup\{\|x\| : x \in R\}$. Our proof may be adapted to show that, for $p \leq p_0$, the probability that K has a subgraph lying in $\mathcal{E}$ and containing O with radius greater than r is at most

$$\exp\left[-\frac{\alpha r}{(\log r)^\sigma}\right]$$

(for suitable constants $\alpha = \alpha(p) > 0$ and $\sigma > 0$). An improvement to this bound is derived (using another method) in [4].

3 Proof of Theorem 2

The purpose of this section is to prove Theorem 2 of the preceding section. We shall start by briefly describing the ideas of the proof. The most important step will be to show that if we consider a certain sequence of cuboid blocks in $\mathbb{R}^3$ having identical shapes but varying sizes, then the probability that such a block B is 'spanned' in the horizontal direction by a disc lying in $[Q]^{\{1/4\}} \cap B$ which 'separates' the top and bottom faces is non-decreasing in the size of B, for p sufficiently close to unity. Figure 4 illustrates the informal notion of a disc spanning a block; an important part of the the proof will be to formulate an appropriate formal definition. It will be deduced that for a certain sequence of disjoint nested cubic 'shells' enclosing the point $(1/2, 1/2, 1/2)$, the probability that a shell contains a sphere separating its inside and outside is non-decreasing in the size of the shell. It will follow that this point is enclosed by some sphere almost surely.

Our approach is to build a large block from similar blocks of a smaller size, and use discs spanning the smaller blocks to construct a disc spanning the larger

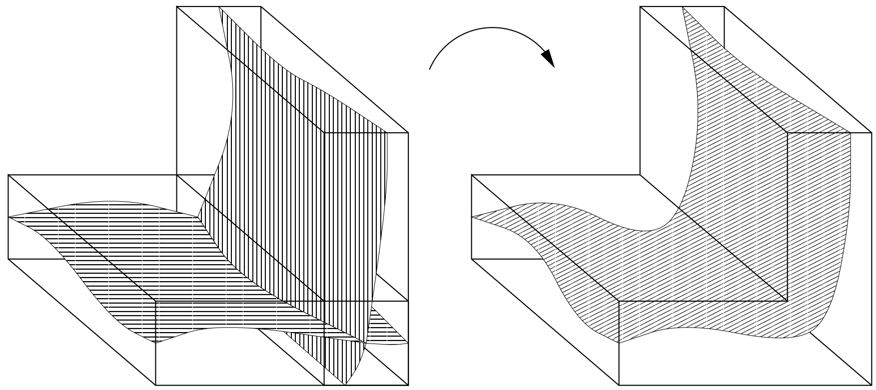

Figure 5: The basic topological result.

block. We may arrange the construction so that (for some M) there are $2M$ smaller blocks arranged in M pairs, such that the larger block is spanned by a disc provided at least one smaller block from each pair is spanned by a disc. Thus if each of the smaller blocks is spanned by a disc with probability $1-\epsilon$ (where ϵ is small), the probability for the larger block is approximately $1-M\epsilon^2$, which is greater than $1-\epsilon$ provided ϵ is sufficiently small. This approach is related to some of the techniques used in [1], but owing to the topological requirement that the surfaces involved should be discs, more care is required than in [1].

The basic topological result which we shall require is that if we have two suitable overlapping blocks, spanned by discs in the horizontal and vertical directions, then we can construct a new disc spanning the compound L-shaped volume (see Figure 5). It might appear that such a disc can always be found as a subset of the union of the two original discs, but this is false. For a counterexample, suppose as in the diagram on the left of Figure 6 that both discs are basically 'flat' and intersect in a line, except that the horizontal disc has a 'finger' protruding from one side, the end of which crosses the vertical disc, intersecting it in a loop. It is easily seen that the union of these two discs does not contain a disc with the required properties. However, in this case, we may find a disc with the required properties by passing to a neighbourhood of the union of the two discs (which may be made sufficiently small as to remain within $[Q]^{\{1/4\}}$) and then taking a subset of this neighbourhood. In this way we may 'cut off' of the end of the finger, replacing the cut end with a disc lying close to the vertical disc (see Figure 6). In effect, we used a part of the vertical disc twice. It turns out that this argument may be applied in the general case, as we shall see in Proposition 3. It is for this reason that we need to use $[Q]^{\{1/4\}}$ rather than $[Q]$.

We remark that it would be possible to reformulate our proof of the inequality $p_{\mathrm{e}}(\mathcal{E})>0$ in a way which did not involve plaquettes, referring instead simply to discs and spheres lying in $\mathbb{R}^3 \setminus [K]$.

The proof of Theorem 2 depends on two topological results, Proposition 3

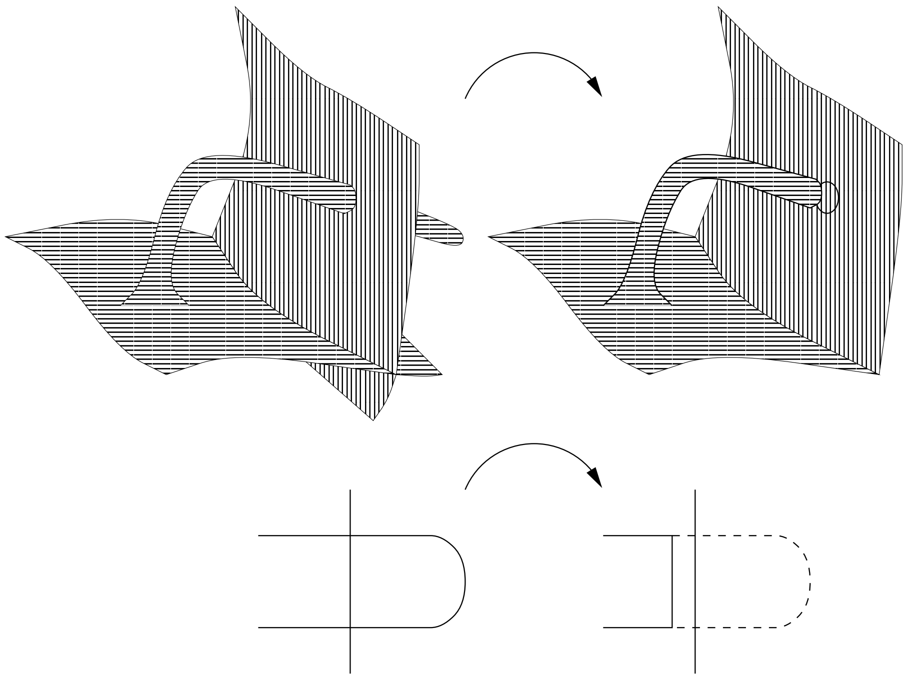

Figure 6: Dealing with a 'finger'. In the lower pair of diagrams, a part of the region seen in the upper pair of diagrams is viewed in cross-section.

and Proposition 4, which we shall state in this section, but whose proofs are postponed until Section 4. The first of these results, Proposition 3, allows us to construct a larger disc from two overlapping discs as described above. The proof employs fairly standard topological techniques and is essentially a more general version of the argument described above. The second, Proposition 4, is a similar result which allows a sphere to be constructed from six overlapping discs.

The first step is to give a precise definition of a disc 'spanning' a cuboid block. It will be convenient to do this in within a slightly more general framework, and for this we shall require some new definitions.

Let S be a sphere. By a **cycle on** S we mean a finite sequence $(A_1, \ldots, A_r)$ of at least three discs lying in S whose interiors are pairwise disjoint, and such that each of $A_1 \cap A_2$, $A_2 \cap A_3, \ldots, A_r \cap A_1$ is a path. By convention we regard the cycle $(A_1, A_2, \ldots, A_r)$ as being equal to each of $(A_2, \ldots, A_n, A_1)$ and $(A_n, \ldots, A_2, A_1)$. An example of a cycle is illustrated in Figure 7. If $\mathcal{A} = (A_1, \ldots, A_r)$ and $\mathcal{B} = (B_1, \ldots, B_{t_r})$ are two cycles such that $A_1 \supseteq B_1 \cup \cdots \cup B_{t_1}$, $A_2 \supseteq B_{t_1+1} \cup \cdots \cup B_{t_2}$, $\ldots$, $A_r \supseteq B_{t_{r-1}+1} \cup \cdots \cup B_{t_r}$ (for some $t_1, \ldots, t_r$) then we say that $\mathcal{B}$ is a **refinement** of $\mathcal{A}$. By a loop **around** H we mean a loop which is a union of the form $\nu_1 \cup \{x_{12}\} \cup \nu_2 \cup \{x_{23}\} \cup \ldots \cup \nu_n \cup \{x_{n1}\}$ where ν_i is the interior of an arc, ν_i lies in the interior of C_i, and x_{ij} is a point lying in the interior of the arc $C_i \cap C_j$. Note that if $\mathcal{B}$ is a refinement of $\mathcal{A}$ then any loop around $\mathcal{B}$ is also a loop around $\mathcal{A}$. A loop around a cycle is illustrated in Figure 7.

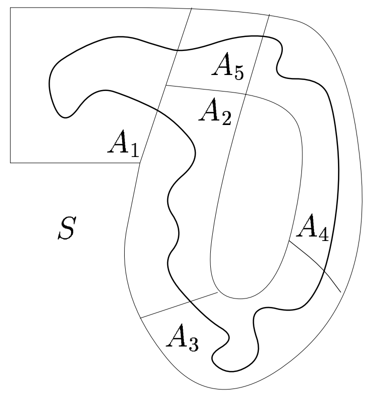

Figure 7: A cycle on S and a loop around the cycle.

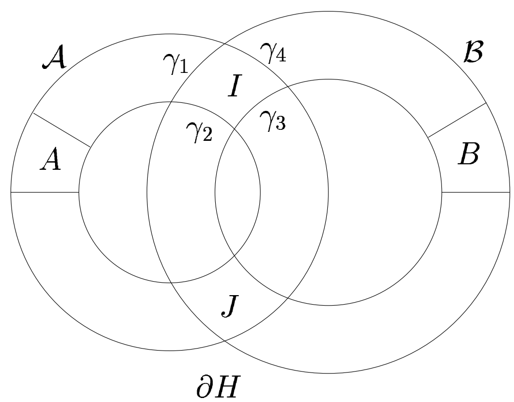

Figure 8: The cycles $\mathcal{A}$ and $\mathcal{B}$ in Proposition 3.

Let H be a ball, and let $\mathcal{A}$ be a cycle on ∂H (in particular, one should keep in mind the case where H is a cuboid, and $\mathcal{A}$ consists of rectangles covering the four vertical faces of ∂H, as illustrated in Figure 9). A disc **in H across $\mathcal{A}$** is a disc $D \subseteq H$ with $\partial D = D \cap \partial H$ and such that ∂D is a loop around $\mathcal{A}$.

Let H be a ball and suppose that $\mathcal{A} = (A_1, \ldots, A_m)$ and $\mathcal{B} = (B_1, \ldots, B_n)$ are two cycles on ∂H. We wish to describe the situation illustrated in Figure 8. Thus we say that $\mathcal{A}$ and $\mathcal{B}$ are **compatible** if the following statements hold. The discs $A_1, \ldots, A_m, B_1, \ldots, B_n$ have pairwise disjoint interiors, except that there exist discs I and J such that $I = A_r = B_u$ and $J = A_t = B_w$ for some r, t, u, w. Furthermore, ∂I is the union of four paths $\gamma_1, \ldots \gamma_4$ which intersect only at their end points in numerical order, such that $\gamma_1 = I \cap A_{r-1}$, $\gamma_3 = I \cap A_{r+1}$, $\gamma_2 = I \cap B_{u-1}$ and $\gamma_4 = I \cap A_{u+1}$, so that we might say that $\mathcal{A}$ and $\mathcal{B}$ 'cross' at I. We also require that the corresponding condition also holds at J.

Suppose $\mathcal{A}$ and $\mathcal{B}$ are compatible, and let A, B be members of $\mathcal{A}, \mathcal{B}$ respectively which are not equal to I or J. We define $\mathcal{A} \vee \mathcal{B}$ to be the cycle on ∂H formed by joining together in the obvious way the subsequence of $\mathcal{A}$ starting at I including A and ending at J, and the subsequence of $\mathcal{B}$ starting at J including B and ending

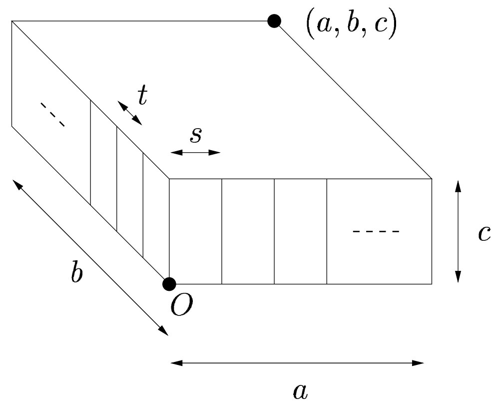

Figure 9: The cycle $\mathcal{H}_{s,t}(a,b,c)$.

at I. (This is a slight abuse of notation since the definition depends on the choice of A and B). Thus if $A = A_s$ and $B = B_v$ where $r < s < t$ and $u < v < w$, we have $\mathcal{A} \vee \mathcal{B} = (I, A_{r+1}, \ldots, A_s, \ldots, A_{t-1}, J, B_{w-1}, \ldots, B_v, \ldots, B_{u+1})$.

Proposition 3 *Let H be a ball and suppose that $\mathcal{A}$ and $\mathcal{B}$ are compatible cycles on ∂H, and let I and J be as above. Choose A and B and define $\mathcal{A} \vee \mathcal{B}$ accordingly. If X is a disc in H across $\mathcal{A}$, and Y is a disc in H across $\mathcal{B}$, then for any $\epsilon > 0$ there exists a disc in H across $\mathcal{A} \vee \mathcal{B}$ which lies in $(X \cup Y)^{\{\epsilon\}} \cap H$.*

See Section 4 for a proof.

We define the **block** $H(a,b,c;d,e,f) = [a,d] \times [b,e] \times [c,f] \subseteq \mathbb{R}^3$; we also write $H(a,b,c) = H(0,0,0;a,b,c)$. If s divides a and t divides b, we define the cycle $\mathcal{H}_{s,t}(a,b,c)$, which we will write as $\mathcal{H}_{s,t}$ when there is no risk of confusion, to be the cycle on $\partial H(a,b,c)$ consisting of rectangles covering the x-z and y-z faces of the block, where the rectangles have z-dimension c, and x-dimension s or y-dimension t respectively, as in Figure 9. (Here and subsequently the symbols x,y and z are used to refer to the 1,2 and 3 coordinate directions respectively).

Let h be a positive integer. We define the blocks

$$\begin{aligned}
F_1(h) &= H(-2h,-2h,-2h;2h,2h,-h) \\
F_2(h) &= H(-2h,-2h,-2h;2h,-h,2h) \\
F_3(h) &= H(-2h,h,-2h;2h,2h,2h) \\
F_4(h) &= H(-2h,-2h,-2h;-h,2h,2h) \\
F_5(h) &= H(h,-2h,-2h;2h,2h,2h) \\
F_6(h) &= H(-2h,-2h,h;2h,2h,2h)
\end{aligned}$$

Thus $F_1, \ldots, F_6$ form the six overlapping 'faces' of a cubic shell of side-length $4h$ and thickness h (see Figure 10). Define $U(h) = F_1 \cup \cdots \cup F_5$ and $S(h) = F_1 \cup \cdots \cup F_6$. Since $F_1, \ldots, F_6$ are all congruent to $H(4h,4h,h)$, we may define

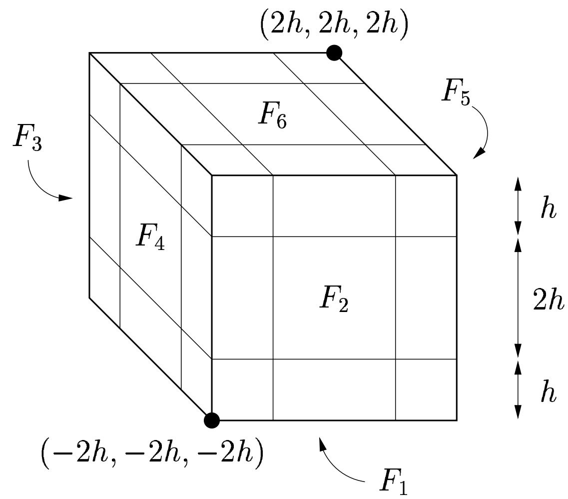

Figure 10: The faces $F_i(h)$.

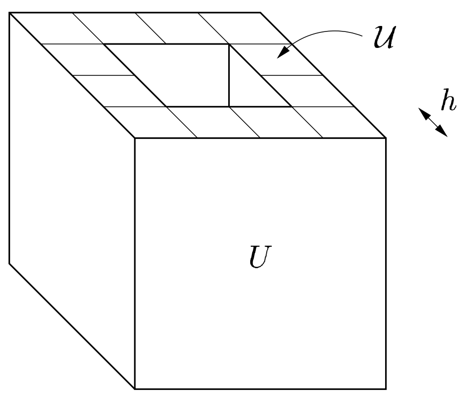

Figure 11: The set U and the cycle $\mathcal{U}$.

$\mathcal{F}_i(h)$ to be the cycle on ∂F_i corresponding to $\mathcal{H}_{h,h}(4h, 4h, h)$. Also let $\mathcal{U}$ be the cycle on ∂U consisting of 12 h-by-h squares covering the annulus face of U in the x-y plane. Figure 11 is an illustration of U and $\mathcal{U}$.

Our second topological result is similar to the first; again, a proof will be given in Section 4.

Proposition 4 *Let h be a positive integer. If X is a disc in F_6 across $\mathcal{F}_6$, and Y is a disc in U across $\mathcal{U}$, then for any $\epsilon > 0$ there exists a sphere lying in $(X \cup Y)^{\{\epsilon\}} \cap S$ with the origin in its inside.*

We shall now make use of the preceding results in the context of the plaquette model. For any compact set $C \subseteq \mathbb{R}^3$ (in particular for C a block), define the set of plaquettes

$$\mathbb{P}(C) = \{u \in \mathbb{P} : [u] \subseteq C\ ,\ [u] \not\subseteq \partial C\},$$

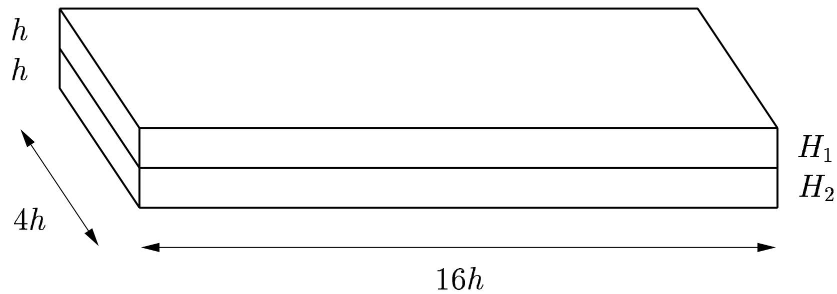

Figure 12: The construction for Lemma 6.

and the random subset of $\mathbb{R}^3$

$$W(C) = [Q \cap \mathbb{P}(C)]^{\{1/4\}} \cap C.$$

If C is a ball and $\mathcal{A}$ is a cycle on ∂C, let $D(C;\mathcal{A})$ be the event that $W(C)$ contains a disc in C across $\mathcal{A}$. Note that $D(C;\mathcal{A})$ is an increasing cylinder event defined on the finite set of plaquettes $\mathbb{P}(C)$. In particular, we write

$$\Pi_q(a,b,c;s,t) = P_q^{\mathbb{P}}(D(H(a,b,c);\mathcal{H}_{s,t})).$$

Lemma 5 *We have*

$$\Pi_q(32,8,2;2,2) \geq q^{256}.$$

PROOF The event $D(H(32,8,2);\mathcal{H}_{2,2})$ occurs if all $32 \times 8 = 256$ x-y plaquettes of $\mathbb{P}(H(32,8,2))$ lie in Q. □

Lemma 6 *For any positive integer h we have*

$$\Pi_q(16h,4h,2h;4h,2h) \geq 1-(1-\Pi_q(16h,4h,h;h,h))^2.$$

PROOF We shall show that $\Pi_p(16h,4h,2h;h,h) \geq 1-(1-\Pi_p(16h,4h,h;h,h))^2$; the result then follows because $\mathcal{H}_{h,h}$ is a refinement of $\mathcal{H}_{4h,2h}$.

The construction is illustrated in Figure 12. Consider the two congruent blocks $H_1 = H(16h,4h,h)$ and $H_2 = H(0,0,h;16h,4h,2h)$; note that $H_1 \cup H_2 = H(16h,4h,2h)$. We write $\mathcal{H}_1 = \mathcal{H}_{h,h}(16h,4h,h)$, and we write $\mathcal{H}_2$ for the corresponding cycle on H_2, $\mathcal{H}_2 = \mathcal{H}_{h,h}(16h,4h,h)+(h,0,0)$. Any disc in H_1 across $\mathcal{H}_1$ is also a disc in $H_1 \cup H_2$ across $\mathcal{H}_{h,h}(16h,4h,2h)$, and so also is any disc in H_2 across $\mathcal{H}_2$. The result now follows from the observation that $\mathbb{P}(H_1)$ and $\mathbb{P}(H_2)$ are disjoint, so $D(H_1;\mathcal{H}_1)$ and $D(H_2;\mathcal{H}_2)$ are independent; the right-hand side of the inequality is the probability of their union. □

Lemma 7 *For any positive integer h we have*

$$\begin{array}{rrcl} & \Pi_q(64h,16h,4h;4h,4h) & \geq & \Pi_q(16h,4h,2h;4h,2h)^{61}, \\ \textit{and} & \Pi_q(16h,16h,4h;4h,4h) & \geq & \Pi_q(16h,4h,2h;4h,2h)^{11}. \end{array}$$

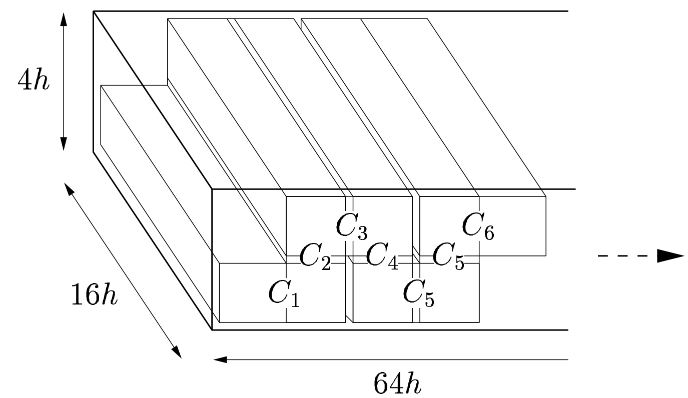

Figure 13: The construction for Lemma 7.

PROOF The construction is illustrated in Figure 13. Define the following sequence of congruent overlapping blocks in $H(64h, 16h, 4h)$.

$$\begin{aligned}
C_1 &= H(0,0,0;4h,16h,2h)\\
C_2 &= H(2h,0,0;4h,16h,4h)\\
C_3 &= H(2h,0,2h;6h,16h,4h)\\
C_4 &= H(4h,0,0;6h,16h,4h)\\
C_5 &= H(4h,0,0;8h,16h,2h)\\
&\vdots\\
C_{61} &= H(60h,0,0;64h,16h,2h)
\end{aligned}$$

Let $\mathcal{C}_1$ be the cycle $\mathcal{H}_{2h,4h}(4h,16h,2h)$ on ∂C_1. Since the C_i are all congruent, let $\mathcal{C}_i$ be the corresponding cycle on ∂C_i for each i. We shall show that $D(H(64h,16h,4h);\mathcal{H}_{4h,4h})$ occurs provided $\bigcap_{i=1}^{61} D(C_i;\mathcal{C}_i)$ occurs, and the first required inequality then follows from the FKG inequality (see [3], page 27). The proof of the second inequality is identical except that we take only the first 11 of the C_i.

Our approach is to make repeated use of Proposition 3 as follows. We first show that provided $D(C_1;\mathcal{C}_1)$ and $D(C_2;\mathcal{C}_2)$ occur, for a suitable cycle $\mathcal{C}_1 \vee \mathcal{C}_2$ on $\partial(C_1 \cup C_2)$, the event $D(C_1 \cup C_2; \mathcal{C}_1 \vee \mathcal{C}_2)$ occurs. We then apply the proposition again to $\mathcal{C}_1 \vee \mathcal{C}_2$ and $\mathcal{C}_3$ to show that provided $D(C_1 \cup C_2; \mathcal{C}_1 \vee \mathcal{C}_2)$ and $D(C_3;\mathcal{C}_3)$ occur then for a suitably chosen cycle $\mathcal{C}_1 \vee \mathcal{C}_2 \vee \mathcal{C}_3$, the event $D(C_1 \cup C_2 \cup C_3; \mathcal{C}_1 \vee \mathcal{C}_2 \vee \mathcal{C}_3)$ occurs. We continue until we have a disc in $\bigcup_{i=1}^{61} C_i$ across a cycle $\bigvee_{i=1}^{61} \mathcal{C}_i$. This is also a disc in $H(64h,16h,4h)$ across $\bigvee_{i=1}^{61} \mathcal{C}_i$, and this latter cycle can be chosen so that it is a refinement of $\mathcal{H}_{4h,4h}(64h,16h,4h)$, so the result will follow.

We shall describe in detail only the first step of the construction, the others being similar. See Figure 14 for an illustration. Suppose that $D(C_1;\mathcal{C}_1)$ and $D(C_2;\mathcal{C}_2)$ occur. Then since $\mathcal{C}_1$ and $\mathcal{C}_2$ are also cycles on $\partial(C_1 \cup C_2)$, the events $D(C_1 \cup C_2;\mathcal{C}_1)$ and $D(C_1 \cup C_2;\mathcal{C}_2)$ occur. Now let X and Y be two discs 'demon-

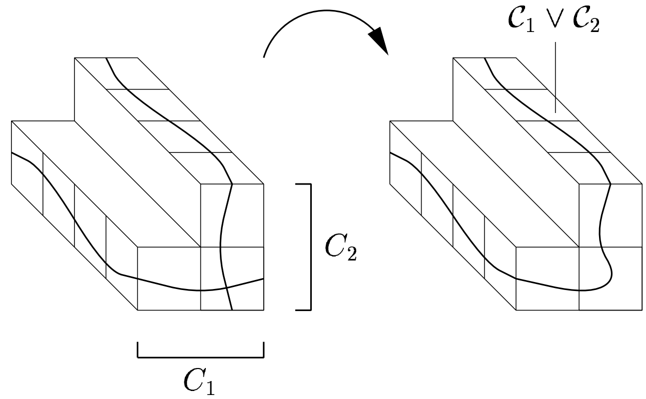

Figure 14: The first step in the proof of Lemma 7.

strating' these two events, and let ϵ be half the distance between the compact sets $X \cup Y$ and $(C_1 \cup C_2) \setminus W(C_1 \cup C_2)$ (ϵ is positive since these sets are disjoint). The cycles $\mathcal{C}_1$ and $\mathcal{C}_2$ are compatible, so we may now apply Proposition 3, with (in the notation of that proposition) $H = C_1 \cup C_2$, $\mathcal{A} = \mathcal{C}_1$, $\mathcal{B} = \mathcal{C}_2$, and $\mathcal{C}_1 \vee \mathcal{C}_2$ being the cycle which consists of four rectangles on each of the extremal faces of $C_1 \cup C_2$ in the negative x and positive z directions, together with three $2h$ by $2h$ squares on each of the two 'L'-shaped x-z faces (see Figure 14). We obtain a disc across this cycle which, by the choice of ϵ, lies in $W(C_1 \cup C_2)$. We now continue as described above. □

Lemma 8 *For any positive integer h we have*

$$P_q(D(U(h); \mathcal{U}(h))) \geq \Pi_q(4h, 4h, h; h, h)^5.$$

Proof The result may be proved in a similar way to the previous lemma, by showing that $D(U; \mathcal{U})$ occurs provided $\bigcap_{i=1}^5 D(F_i; \mathcal{F}_i)$ occurs. Again we do this by 'adding' one block at a time (in the order $F_1, \ldots, F_5$) using Proposition 3. The cycles at intermediate steps are chosen so as to ensure that $\bigvee_{i=1}^5 \mathcal{F}_i = \mathcal{U}$. □

We are now in a position to prove the main result of this section.

Proof of Theorem 2 For given q, and n a nonnegative integer, define

$$\pi_n = \Pi_q(32 \cdot 4^n, 8 \cdot 4^n, 2 \cdot 4^n; 2 \cdot 4^n, 2 \cdot 4^n).$$

By Lemma 6 and the first inequality of 7 we have

$$\begin{aligned} \pi_{n+1} &\geq f(\pi_n) \text{ where} \\ f(u) &= (1 - (1-u)^2)^{61}. \end{aligned}$$

Since $f(1) = 1$ and $f'(1) = 0$, there exists some $0 < w < 1$ such that if $u \geq w$ then $f(u) \geq w$. Let $q_0 = w^{1/256}$, so that if $q \geq q_0$ then by Lemma 5 we have

$\pi_0 \geq w$, hence $\pi_n \geq w$ for all n. (An explicit calculation shows that we may take $q_0 = 1 - 1/15616$). Hence by the second inequality of Lemma 7 we also have for all n,

$$\Pi_q(8 \cdot 4^n, 8 \cdot 4^n, 2 \cdot 4^n; 2 \cdot 4^n, 2 \cdot 4^n) \geq w.$$

Let A_n be the event that $W(S(2 \cdot 4^n))$ contains a sphere with origin (or equivalently the point $(1/2, 1/2, 1/2)$) in its inside. Then by Proposition 4, A_n occurs provided $D(U; \mathcal{U})$ and $D(F_6; \mathcal{F}_6)$ occur. By Lemma 8 and the FKG inequality, this occurs with probability at least $\Pi_q(8 \cdot 4^n, 8 \cdot 4^n, 2 \cdot 4^n; 2 \cdot 4^n, 2 \cdot 4^n)^6$, so if $q \geq q_0$ then $P_q(A_n) \geq w^6 > 0$ for all n. Hence since the events A_n are independent we have $P_q(A_n$ occurs for some $n) = 1$, completing the proof. □

We remark that the constant $p_0 = 1/15616$ is almost certainly not the best possible; that is, our argument could be made to work for some larger p_0. However, it seems unlikely that any value so obtained would be large enough to provide a useful numerical lower bound on p_{e}.

4 Proofs of topological results

In this section we shall prove the two topological results, Propositions 3 and 4. We shall employ the method of 'surgery', and we shall make frequent use of several standard topological theorems (such as the Schönflies Theorem; [7], page 19), which are intuitively plausible but non-trivial to prove. We shall present the arguments in a slightly informal way, in the sense that we shall not generally make explicit references to such theorems; these references would be lengthy but uninformative. This approach is common in the topology literature - see for example [7], Chapter 2. It is hoped that the arguments presented in this way will serve a two-fold purpose: to the reader unfamiliar with the relevant topological ideas, they should provide an intuitively plausible justification of the results; while to the topological expert they should constitute a rigorous proof. Detailed justification of the topological steps may be found for example in [8].

The following topological ingredient of the proofs deserves special mention. We shall make extensive use of *transversality* (or *general position*). Roughly speaking, this is the assertion that if we have a number of intersecting simplicial complexes in $\mathbb{R}^3$, and if we are allowed to deform them by a small amount, then we may assume that all their intersections are of 'canonical' type, with no 'coincidences' such as three paths in a surface having a common intersection, or two surfaces touching tangentially. For more details see for example [8].

PROOF OF PROPOSITION 3 Without loss of generality (by deforming X and Y by a small amount if necessary within $(X \cup Y)^{\{\epsilon\}} \cap H$), we may assume that X and Y intersect only transversely. Hence $X \cap Y$ is a 1-dimensional manifold with boundary $\partial(X \cap Y) = \partial X \cap \partial Y$. By considering the intersections of ∂X and ∂Y

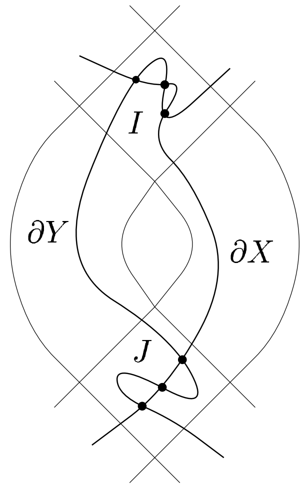

Figure 15: The intersections of ∂X and ∂Y.

with I and J, we see that $\partial X \cap \partial Y$ must be the union of an odd number of points in the interiors of each of I and J (see Figure 15). Thus $X \cap Y$ is the union of finitely many components of the following types (the finiteness follows from the fact that X and Y are simplicial).

(i) Loops

(ii) Arcs with both ends in one of I or J

(iii) Arcs with one end in each of I and J.

We shall show that by making alterations to Y we may 'remove' all components of types (i) and (ii).

Consider $X \cap Y$ as a subset of X (see Figure 16 for an illustration). Note that $X \cap I$ and $X \cap J$ are a pair of disjoint arcs lying in ∂X. Suppose that there is at least one component of type (i), and let α be an innermost such component on X, bounding a disc D in X. The requirement that α be innermost ensures that the interior of D has no intersections with Y. We form a new surface from Y as follows. Remove a small annulus neighbourhood of α in Y, and add two slightly shifted copies of D with their boundaries coinciding with those of the annulus, one on each side of D (see Figure 17). Let Y' be the resulting surface. We have in effect 'cut' Y along a loop and glued two discs to the cut edges. Hence Y' must have two disjoint components: a disc with the same boundary as Y, and a sphere. Let Y'' be the disc component. Now Y'' is a disc across $\mathcal{B}$, and $X \cap Y''$ has strictly fewer components than $X \cap Y$, and provided the alteration was sufficiently 'small', Y'' lies in $(X \cup Y)^{\{\epsilon\}} \cap H$. We may repeat this construction until all components of type (i) have been removed.

Now assume that there are no components of type (i), but that there is at least one component of type (ii). Let β be such a component closest to ∂X on

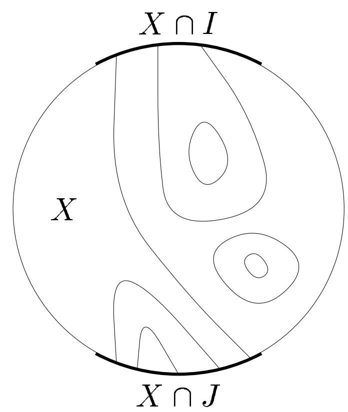

Figure 16: Components of $X \cap Y$.

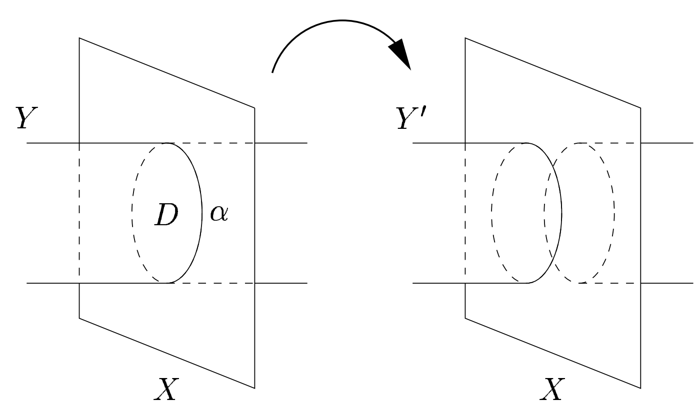

Figure 17: Removing a loop component of $X \cap Y$.

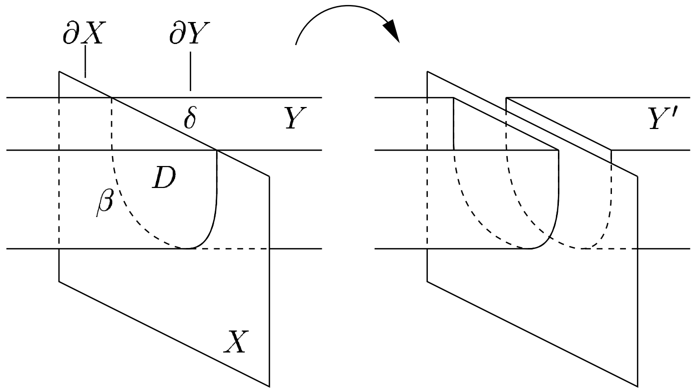

Figure 18: Removing a path component of $X \cap Y$.

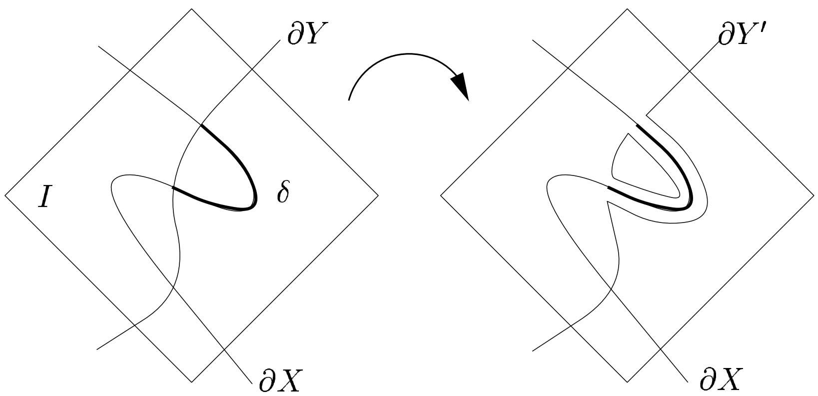

Figure 19: The effect on ∂Y of removing a path component of $X \cap Y$.

X, and suppose without loss of generality that both ends lie in I. There exists a path $\delta \subseteq \partial X \cap I$ joining the two ends of β, which does not intersect ∂Y. Let D be the disc in X bounded by $\beta \cup \delta$ (see Figure 18). We alter Y in a way similar to before; remove a small neighbourhood of β (which is a long thin disc), and add two slighlty shifted copies of D, as illustrated in Figure 18. Let Y' be the resulting surface. Since we have in effect cut Y along a path joining two points on the boundary, Y' must be the union of two disjoint discs, and $\partial Y'$ is equal to ∂Y with two short paths removed and two copies of δ added. Since ∂Y and $\partial Y'$ differ only in a small neighbourhood of δ, by considering their intersection with I, it may be seen that one component of $\partial Y'$ must be a loop around $\mathcal{B}$ (see Figure 19). Let Y'' be the component of Y' bounded by this component of $\partial Y'$. Now $X \cap Y''$ has fewer components than $X \cap Y$, and Y'' lies in $(X \cup Y)^{\{\epsilon\}} \cap H$ provided the alterations were small enough.

After repeatedly applying the arguments above, we may assume that $X \cap Y$ has no components of type (i) or (ii). Since each of I and J must contain an odd number of points of $\partial(X \cap Y)$, there must be at least one component of type (iii). Consider $X \cap Y$ as a subset of X, and let γ be the component of type (iii) nearest

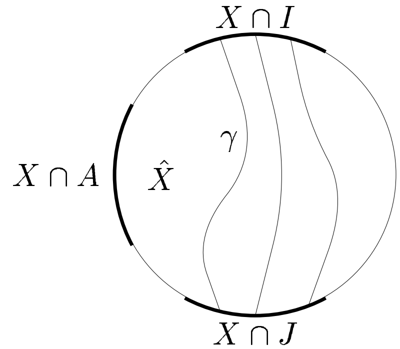

Figure 20: The choice of the path γ.

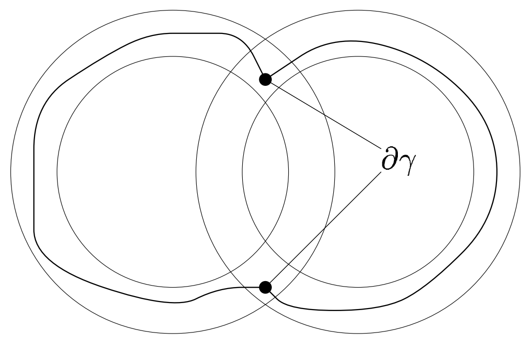

Figure 21: The boundary of $\hat{X} \cap \hat{Y}$.

to $\partial X \cap A$ on X (see Figure 20). Now γ cuts X into two discs; we let $\hat{X}$ be the one which contains $\partial X \cap A$. Similarly γ cuts Y into two discs; we let $\hat{Y}$ be the one which contains $\partial Y \cap B$. Now $\hat{X} \cap \hat{Y}$ is precisely γ, so $\hat{X} \cup \hat{Y}$ is a disc; and it is easily seen that its boundary is a loop around $\mathcal{A} \vee \mathcal{B}$ (see Figure 21). $\square$

Proof of Proposition 4 The proof is similar to the above. Let L be the straight line segment path joining the origin to the point $(0, 0, 3h)$ (see Figure 22). Note that any sphere lying in S which intersects L transversely in an odd number of points must have the origin in its inside. Without loss of generality (after deforming by a small amount if necessary), X, Y and L intersect pairwise only transversely, and $X \cap Y \cap L$ is empty. Now L must intersect X in an odd number of points, since we may extend L outside F_6 to a loop which is linked with ∂X (see Figure 23). Note that $X \cap Y$ is a 1-dimensional manifold with no boundary, and hence is a union of finitely many disjoint loops. Consider $X \cap Y$ and $X \cap L$ as (disjoint) subsets of X (see Figure 25). Observe that each point of $X \cap L$ must be enclosed (on X) by at least one loop of $X \cap Y$. Indeed, if not we could find a path in X joining a point on L to a point on ∂X which did not intersect Y, and this would enable us to construct a loop linked with ∂Y but not intersecting Y, a contradiction (see Figure 24).

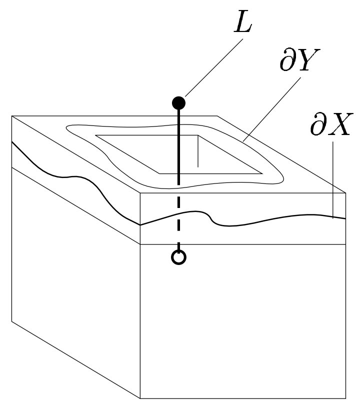

Figure 22: Objects used in the proof of Proposition 4.

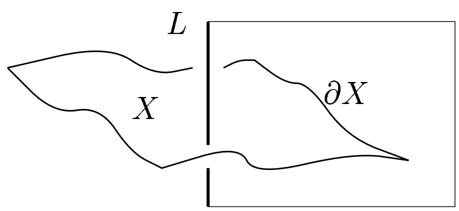

Figure 23: Extending L to a loop which is linked with ∂X.

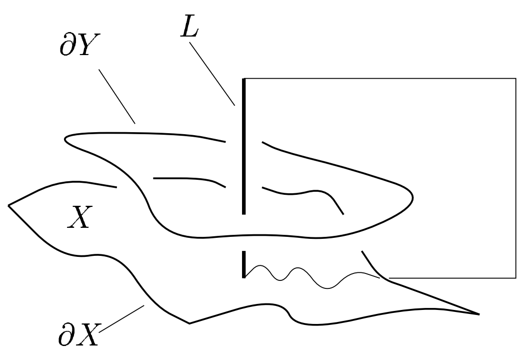

Figure 24: Using a path in X to construct a loop linked with ∂Y.

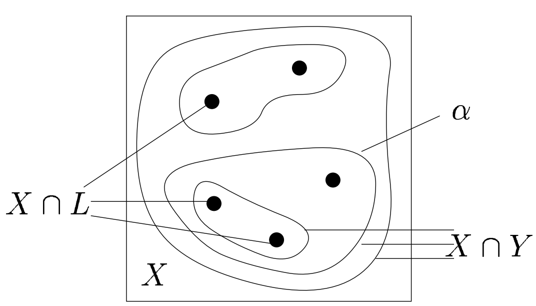

Figure 25: Intersections of X with L and Y.

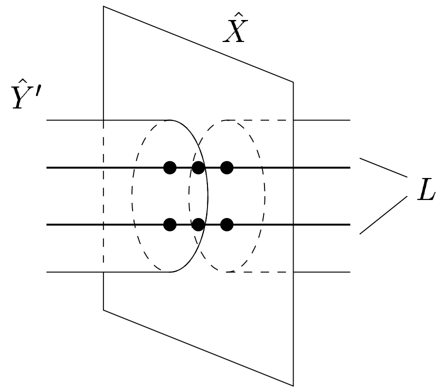

Figure 26: Removing a component of $\hat{X} \cap \hat{Y}$.

By a simple counting argument, it follows from the above observations that there must be some loop of $X \cap Y$ enclosing (on X) an odd number of points of $X \cap L$. Let α be such a loop (see Figure 25). Let $\hat{X}$ be the disc in X bounded by α and let $\hat{Y}$ be the disc in Y bounded by α. Note that the discs $\hat{X}$ and $\hat{Y}$ have the following properties.

(i) $\hat{X}$ and $\hat{Y}$ are subsets of $(X \cup Y)^{\{\epsilon\}} \cap S$

(ii) $\alpha \subseteq \hat{X} \cap \hat{Y}$

(iii) $\hat{X}$ intersects L in an odd number of points

(iv) $\hat{Y}$ intersects L in an even number of points (see note below)

Note that the number of points in (iv) is in fact zero, since $Y \cap L = \emptyset$; in what follows we shall use the statements (i)–(iv) as induction hypotheses, and this number may then become non-zero.

If $\hat{X} \cap \hat{Y}$ consists only of α, then $\hat{X} \cup \hat{Y}$ is a sphere which intersects L in an odd number of points, so we are done. Otherwise we shall show that by altering $\hat{Y}$ we may strictly reduce the number of components of $\hat{X} \cap \hat{Y}$ while ensuring that conditions (i)–(iv) still hold, so the result will follow by induction.

Suppose β is a loop of $\hat{X} \cap \hat{Y}$ innermost on $\hat{X}$. We remove β as in the previous proof by removing a small annulus neighbourhood of β in $\hat{Y}$ and replacing it with two discs to obtain a new surface $\hat{Y}'$. Let y be the number of points of $\hat{Y} \cap I$, and let b be the number of points of $X \cap L$ enclosed by β on X; then it is easily seen (see Figure 26) that $\hat{Y}' \cap I$ consists of exactly $y + 2b$ points, an even number by condition (iv). As in the previous proof, the surface $\hat{Y}'$ must have two components; a sphere, and a disc with boundary α. If the sphere has an odd number of intersections with L, we are done; otherwise let $\hat{Y}''$ be the disc. Now it is easily seen that conditions (i)–(iv) hold with $\hat{Y}''$ in place of $\hat{Y}$. $\square$

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