Entanglement and Rigidity in Percolation Models

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Abstract

We review recent progress and open problems in entanglement and rigidity percolation.

1 Introduction

Percolation models are of interest both for the wide range of their physical applications and for the mathematical challenges which they present. In the basic model, edges of the d-dimensional cubic lattice are independently deleted with a fixed probability, and the connectivity properties of the resulting graph are studied. One natural way to extend this model is to consider other graph properties in place of connectivity. Two such properties of particular interest are entanglement and rigidity. Loosely speaking, the meaning of these terms is as follows. A graph in three-dimensional space is entangled if it cannot be 'pulled apart' when the edges are regarded as physical connections made of elastic. A graph is rigid if it cannot be 'deformed' when the edges are regarded as solid rods which can pivot at the vertices. These intuitive notions will be formalised later.

Entanglement and rigidity in percolation are of interest for several reasons. Firstly, they have important physical applications; specifically, entanglement is relevant to the study of polymers in solution, and rigidity is relevant to the study of glassy materials; see [16], [15] for details. Secondly, certain standard percolation results appear to require novel methods of proof in the case of entanglement or rigidity. Thirdly, there is the possibility of new types of behaviour not found in the connectivity case; in particular it appears that boundary conditions may play a non-trivial role.

The purpose of this article is to provide an introduction to the subject with a minimum of technical detail, and to describe known results and open problems. For more detail the reader is referred to the following papers. For entanglement, the most comprehensive treatment is in [7]; existence of a non-trivial phase transition is proved in [13] and [1]; uniqueness of the infinite cluster is proved in [9]; details of the physical applications may be found in [16]; other review-type material appears in [4] and [5]. For rigidity, the model is introduced [12]; uniqueness is proved in [8]; results on boundary conditions are proved in [11]; physical applications are described in [15]. Details of the mathematical theory of graph rigidity may be found in [3].

This article is organised as follows. The percolation model and basic notation are introduced in Section 2. Entanglement and rigidity are defined in Sections 3 and 4 respectively. The existence of phase transitions is discussed in Section 5. In Section 6 we give results on uniqueness of infinite clusters, and in Section 7 we discuss results on boundary conditions.

2 Percolation

The percolation model is defined as follows. Let L be an infinite connected graph. For our purposes, L will be either the d-dimensional cubic lattice $\mathbb{Z}^d$ (with nearest-neighbour edges), or the two-dimensional triangular lattice $\mathbb{T}$. ($\mathbb{T}$ is a planar graph with six equilateral triangles meeting at each vertex). Given a parameter $p \in [0, 1]$, each edge of L is declared **open** with probability p, and **closed** otherwise, with distinct edges receiving independent declarations. The resulting probability measure is denoted P_p, and a typical configuration of open and closed edges is denoted ω.

Let $\mathcal{A}$ be a set of subgraphs of L. We think of $\mathcal{A}$ as being the set of all graphs having a certain property. The motivating example is

$$\mathcal{C} = \{\text{all connected subgraphs of } L\}.$$

We shall be concerned also with

$$\mathcal{E} = \{\text{all entangled subgraphs of } L\}$$

and

$$\mathcal{R} = \{\text{all rigid subgraphs of } L\}.$$

(We shall give precise definitions later). By an $\boldsymbol{\mathcal{A}}$**-graph** we mean a graph lying in $\mathcal{A}$, and by an $\boldsymbol{\mathcal{A}}$**-subgraph** of a graph we mean a subgraph which is an $\mathcal{A}$-graph. An $\boldsymbol{\mathcal{A}}$**-component** of a graph G is a maximal $\mathcal{A}$-subgraph of G. Thus the $\mathcal{C}$-components of a graph are simply the connected components in the usual sense. An $\boldsymbol{\mathcal{A}}$**-cluster** is an $\mathcal{A}$-component of the (random) graph of all open edges of L.

The usual theory of percolation is concerned with the study of $\mathcal{C}$-clusters. For a distinguished vertex O of L, we define

$$\theta(p) = P_p(\text{there is an infinite } \mathcal{C}\text{-cluster containing } O),$$

and the critical point

$$p_c = \sup\{p : \theta(p) = 0\}.$$

Details of percolation theory may be found in [6].

3 Entanglement

Our study of entanglement percolation is restricted to the case $L = \mathbb{Z}^3$. As in the theory of knots (see [17]), the entanglement of graphs is intrinsically three-dimensional. The choice of the cubic lattice is a matter of convenience, and other choices are possible.

We associate a subgraph of $\mathbb{Z}^3$ with a subset of $\mathbb{R}^3$ in the following natural way: each vertex is a point in $\mathbb{R}^3$, and each edge corresponds to a straight line segment joining its two vertices. We define entanglement first for finite graphs, as follows. A **sphere** is a piecewise-linear subset of $\mathbb{R}^3$ which is homeomorphic to a topological 2-sphere. We say that a sphere S **separates** a set $R \subseteq \mathbb{R}^3$ if R intersects both path-components of $\mathbb{R}^3 \setminus S$, but not S itself. We say that a *finite* subgraph G of $\mathbb{Z}^3$ is **entangled** if it is separated by no sphere. We write

$$\mathcal{E}_F = \{\text{all finite entangled subgraphs of } \mathbb{Z}^3\}.$$

It is clear that any finite connected graph is entangled. The simplest non-connected entangled graph consists of two linked loops of edges. Let e_n be the number of $\mathcal{E}_F$-graphs with exactly n edges containing O. It is an open problem to determine whether e_n grows exponentially with n, or faster.

There are several natural ways to extend the definition of entanglement to infinite graphs. We discuss this further in Section 7 on boundary conditions.

For now, we adopt the following natural definition:

$\mathcal{E} =$

$\{G$: every finite subgraph of G is contained in some $\mathcal{E}_F$-subgraph of $G\}$.

$\mathcal{E}$-components and $\mathcal{E}$-clusters are defined as in Section 2. It may be shown that the edge sets of the $\mathcal{E}$-components of a graph partition the edge set of the graph. We define

$$\eta(p) = P_p(\text{there is an infinite } \mathcal{E}\text{-cluster containing } O),$$

and

$$p_{\mathrm{e}} = \sup\{p : \eta(p) = 0\}.$$

4 Rigidity

Our study of rigidity percolation is restricted to the case $L = \mathbb{T}$, the two-dimensional triangular lattice. Graph rigidity becomes much more complicated in three and higher dimensions (see [3]). In two dimensions, the most obvious choice, $\mathbb{Z}^2$, is of no interest because $\mathbb{Z}^2$ itself is not rigid. The planarity of $\mathbb{T}$ plays an important role in some of the proofs to be discussed.

We define rigidity first for finite graphs. There are several different definitions available, but we shall restrict our attention to so-called generic two-dimensional rigidity. An **embedding** of a finite graph G is an injective map r from the vertex set to $\mathbb{R}^2$. A **motion** of the pair (G, r) is a one-parameter differentiable family of embeddings $(r_t)_{0 \leq t \leq 1}$ of G, containing r, which preserves all edge lengths; that is

$$\|r_t(x) - r_t(y)\| \text{ is constant in } t$$

whenever (x, y) is an edge. A motion is **rigid** if the above statement holds for *every* pair of vertices x, y. The pair (G, r) is said to be **rigid** if all of its motions are rigid motions. It may be shown that for any G, under a natural measure on the set of embeddings, either (G, r) is rigid for almost all r, or (G, r) is not rigid for almost all r (see [3]). We therefore say that G is **rigid** if the former holds, and define

$$\mathcal{R}_F = \{\text{all finite rigid subgraphs of } \mathbb{T}\}.$$

It may be seen that, for example, a triangle is a rigid graph, whereas a square is not. It may be shown that every finite rigid graph is connected.

We extend the definition of rigidity to infinite graphs as follows:

$$\mathcal{R} = \{G : \text{every finite subgraph of } G \text{ is contained in some } \mathcal{R}_F\text{-subgraph of } G\}.$$

$\mathcal{R}$-components and $\mathcal{R}$-clusters defined as in Section 2. It may be shown that the edge sets of the $\mathcal{R}$-components of a graph partition the edge set of the graph, although two different $\mathcal{R}$-components may share a vertex. For a fixed edge e of $\mathbb{T}$ we define

$$\phi(p) = P_p(\text{there is an infinite } \mathcal{R}\text{-cluster containing } e),$$

and

$$p_{\mathrm{r}} = \sup\{p : \phi(p) = 0\}.$$

5 Phase Transitions

The most fundamental result of percolation theory is that for a wide range of graphs L (including $\mathbb{T}$ and $\mathbb{Z}^d$, for $d \geq 2$), we have $0 < p_{\mathrm{c}} < 1$, so that there is a non-trivial phase transition. In this section we discuss corresponding results for entanglement and rigidity.

For entanglement we have the following.

Theorem *For $L = \mathbb{Z}^3$ we have*

$$0 < p_{\mathrm{e}} < p_{\mathrm{c}}.$$

The former inequality is proved in [13]. Such inequalities are usually proved by simple path-counting methods - here it appears that a more complicated argument is required. The approach in [13] is to show that for p sufficiently small there is almost surely a topological sphere which encloses O and which does not intersect any open edges. The topological aspect adds significant complications to the proof. A related open problem is the following. In the 'plaquette percolation model', which is the natural dual to bond percolation on $\mathbb{Z}^3$, is it the case that for sufficiently high density of plaquettes, there is a sphere of plaqettes enclosing O? For more details see [7].

The inequality $p_{\rm e} \le p_{\rm c}$ is immediate from the fact that $\mathcal{C} \subseteq \mathcal{E}$. The corresponding strict inequality follows from an argument in [1]. The key idea is to find a local rule for adding open edges to a configuration (called an enhancement), with the property that it can affect the large-scale connectivity properties of the configuration, but not the large-scale entanglement properties.

An intriguing unsolved problem is to obtain good rigorous bounds on the numerical value of $p_{\rm e}$. At present the only rigorous bounds are as follows. In [13] it is proved that $p_{\rm e} \ge 1/15616$. The technique used to prove $p_{\rm e} < p_{\rm c}$ in [1] may be used to obtain rigorous lower bounds on the difference $p_{\rm c} - p_{\rm e}$, but such bounds are extremely small (of the order 10^{-10}). The numerical value of $p_{\rm c}$ for $\mathbb{Z}^3$ is believed to be approximately 0.249 ([14]), and it easy to obtain the rigorous bound $p_{\rm c} \ge 1/5$ ([6]).

Another open problem is to prove that the size of the $\mathcal{E}$-cluster at O has exponentially decaying tails for $p < p_{\rm e}$. So far the best result in this direction is the following, proved in [7] using an extension of the methods in [13]. For p sufficiently small, the probability that the radius of the $\mathcal{E}$-cluster exceeds r is at most $\exp[-\alpha(p)r/\log r]$, where $\alpha(p) > 0$. Indeed, the logarithm in this expression may be replaced with any iterate of logarithm (for suitable α).

For rigidity we have the following.

Theorem *For $L = \mathbb{T}$ we have*

$$p_{\rm c} < p_{\rm r} < 1.$$

Both inequalities are proved in [12]. The proof of the former uses the technology of [1]. In this case, the idea is to find a rule for removal of open edges (a 'diminishment'), which can affect the connectivity properties but not the rigidity properties of the configuration.

6 Uniqueness

It is a standard result that for $L = \mathbb{Z}^d$ (or $\mathbb{T}$), and any p, there is at most one infinite $\mathcal{C}$-cluster almost surely. In this section we discuss extensions to entanglement and rigidity.

For entanglement, the following is proved in [9].

Theorem *For $L = \mathbb{Z}^3$ and $p > p_{\rm e}$, there is exactly one infinite $\mathcal{E}$-cluster almost surely.*

It is unknown how many infinite $\mathcal{E}$-clusters there are when $p = p_{\mathrm{e}}$; by standard arguments (originating from [18]) the answer is one of 0, 1 or ∞ almost surely.

The proof of the above theorem uses an interesting combination of the idea of 'monotonicity of uniqueness' (as in [10]) with the technology of [2]. The basic idea is as follows. Consider the usual coupling of the percolation model with two different parameters $(p_{\mathrm{e}} <) p_1 < p_2$. Define a new graph as follows. Augment the set of open edges at level p_2 by adding an extra edge between very pair of vertices which are in the same $\mathcal{E}$-cluster at level p_1. The method of [2] may be used to show that this graph has a unique infinite $\mathcal{C}$-cluster. Using ideas from [10], it may be shown that every infinite $\mathcal{E}$-cluster at level p_2 contains an infinite $\mathcal{E}$-cluster at level p_1. Combining these facts yields uniqueness of the infinite $\mathcal{E}$-cluster at level p_2.

For rigidity, the following stronger result is proved in [8].

Theorem *For $L = \mathbb{T}$ and all p, there is at most one infinite $\mathcal{R}$-cluster almost surely.*

(A weaker version of this result was obtained earlier in [12]). The proof in [8] makes use of the planarity of $\mathbb{T}$, and involves 'surrounding circuits'.

7 Boundary Conditions

The study of boundary conditions may be motivated as follows. Instead of considering infinite clusters in the infinite graph L, we concentrate on a large finite portion of L, and consider the limits of quantities of interest as the size of this finite portion increases. The question then arises of how the boundary of the finite portion should be treated, and different choices may in principle give different results. Two extremal choices are 'free' or '0' boundary conditions, in which all external edges are declared closed, and 'wired' or '1' boundary conditions, in which all external edges are declared open. The questions which arise are of interest both mathematically and in the context of applications.

For L equal to $\mathbb{Z}^d$ or $\mathbb{T}$, let B_n be the graph of all vertices and edges within graph-theoretic distance n of O. (B_n takes the form of a hypercube for $\mathbb{Z}^d$, and a hexagon for $\mathbb{T}$). Given a configuration ω, we define new configurations ω_n^i for $i = 0, 1$ as follows. On B_n, ω_n^i agrees with ω, while outside B_n, all edges are declared closed if $i = 0$, and open if $i = 1$. Fix ω, and suppose $\mathcal{A}$

is one of $\mathcal{C}, \mathcal{E}, \mathcal{R}$. We say that a graph G is an $\boldsymbol{\mathcal{A}^0}$**-cluster** (of ω) if it is an increasing limit of a sequence of graphs G_n, where G_n is an $\mathcal{A}$-cluster of ω_n^0. We say that a graph G is an $\boldsymbol{\mathcal{A}^1}$**-cluster** (of ω) if it is a decreasing limit of a sequence of graphs G_n, where G_n is an $\mathcal{A}$-cluster of ω_n^1. An $\mathcal{A}^i$-cluster may be interpreted as 'an $\mathcal{A}$-cluster when boundary condition i is applied'.

In the case of connectivity, it is straightforward to see that boundary conditions have have only trivial effects. The concepts of a $\mathcal{C}$-cluster and a $\mathcal{C}^0$-cluster coincide (for every ω). Turning to $\mathcal{C}^1$-clusters, every *finite* $\mathcal{C}^0$-cluster is also a $\mathcal{C}^1$-cluster, while the union of all infinite $\mathcal{C}^0$-clusters (if there are any) forms a single $\mathcal{C}^1$-cluster. Since in the percolation model (on $\mathbb{Z}^d$ or $\mathbb{T}$) there is at most one infinite $\mathcal{C}$-cluster almost surely, this implies that $\mathcal{C}^0$-clusters and $\mathcal{C}^1$-clusters coincide almost surely.

The situation is more complicated in the case of entanglement. It turns out that the concepts of an $\mathcal{E}$-cluster and an $\mathcal{E}^0$-cluster coincide. $\mathcal{E}^1$-clusters can be interpreted in the following way. Define

$$\mathcal{E}' = \{G : G \text{ is separated by no sphere}\}.$$

We have $\mathcal{E}' \supseteq \mathcal{E}$, but the sets are not equal. The simplest graph in $\mathcal{E}'$ but not in $\mathcal{E}$ consists of two disjoint doubly-infinite paths. It turns out that the concepts of an $\mathcal{E}'$-cluster and an $\mathcal{E}^1$-cluster coincide. In contrast with the connectivity case, there exist configurations for which there is an infinite $\mathcal{E}^1$-cluster but no infinite $\mathcal{E}^0$-cluster (see [7] for examples). The question therefore arises whether such configurations can actually occur in the percolation model.

The above question may be addressed as follows. For $i = 0, 1$ define

$$\eta^i(p) = P_p(\text{there is an infinite } \mathcal{E}^i\text{-cluster containing } O),$$

and

$$p_{\mathrm{e}}^i = \sup\{p : \eta^i(p) = 0\}.$$

(Thus, $\eta^0 = \eta$ and $p_{\mathrm{e}}^0 = p_{\mathrm{e}}$). It is straightforward to show that $\eta^0 \leq \eta^1$, and therefore $p_{\mathrm{e}}^1 \leq p_{\mathrm{e}}^0$. The proof that $p_{\mathrm{e}}^0 > 0$ in [13] also gives $p_{\mathrm{e}}^1 > 0$. The question is whether or not $p_{\mathrm{e}}^0 = p_{\mathrm{e}}^1$. More generally, for what values of p does $\eta^0(p) = \eta^1(p)$ hold? There has been little progress on these questions. In [7] it is proved that $\eta^0(p) = \eta^1(p)$ for p sufficiently close to 1.

There has been more progress on the analogous problems for rigidity percolation on $\mathbb{T}$. Again, the concepts of an $\mathcal{R}$-cluster and an $\mathcal{R}^0$-cluster coincide. As with entanglement (but not connectivity), there exist configurations for which there is an infinite $\mathcal{R}^1$-cluster but no infinite $\mathcal{R}^0$-cluster (see [11]). We define

$$\phi^i(p) = P_p(\text{there is an infinite } \mathcal{R}^i\text{-cluster containing } e),$$

and

$$p_{\mathrm{r}}^i = \sup\{p : \phi^i(p) = 0\}$$

(so that $\phi^0 = \phi$ and $p_{\mathrm{r}}^0 = p_{\mathrm{r}}$).

The following is proved in [11].

Theorem *We have $p_{\mathrm{r}}^0 = p_{\mathrm{r}}^1 (= p_{\mathrm{r}})$. Furthermore, $\phi^0(p) = \phi^1(p)$ except possibly at $p = p_{\mathrm{r}}$.*

As a corollary of this and the uniqueness of the infinite $\mathcal{R}^0$-cluster (Section 6), it may be shown that $\mathcal{R}^0$-clusters and $\mathcal{R}^1$-clusters coincide almost surely, except possibly when $p = p_{\mathrm{r}}$. In other words, boundary conditions have no effect except possibly at the critical point.

Here is a sketch of the ideas behind the proof the above theorem. Define the number of degrees of freedom of a finite graph to be the minimum number of edges which must be added to make the graph rigid. If $\phi^1(p) > \phi^0(p)$ then there is an infinite $\mathcal{R}^1$-cluster, I^1 say, which is not a single $\mathcal{R}^0$-cluster. Furthermore, it may be shown that there is a positive density of edges of $\mathbb{T}$ whose addition would reduce the number of degrees of freedom of (a finite approximation to) I^1. If it were the case that $\phi^1 > \phi^0$ throughout some interval, Russo's formula would imply that I^1 must have (in an appropriate sense) a positive density of degrees of freedom per unit area. But this gives a contradiction, since for a large box B_n, only $O(n)$ edges (around the boundary of B_n) need to be added to make $I^1 \cap B_n$ rigid.

The results mentioned above lead to partial information about the continuity of ϕ^0 and ϕ^1. Specifically, exactly one of the following must hold.

(i) ϕ^0 and ϕ^1 are equal everywhere and continuous everywhere;

(ii) ϕ^0 and ϕ^1 are equal everywhere and continuous except at p_{r}, where both are right-continuous;

(iii) ϕ^0 and ϕ^1 are equal and continuous except at p_{r}, where they are unequal, and ϕ^1 is right-continuous while ϕ^0 is left-continuous.

It is a fascinating unsolved problem to determine which of (i)–(iii) is correct.

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