Local Bootstrap Percolation

Janko Gravner*   Alexander E. Holroyd†

June 16, 2008

Abstract

We study a variant of bootstrap percolation in which growth is restricted to a single active cluster. Initially there is a single *active* site at the origin, while other sites of $\mathbb{Z}^2$ are independently *occupied* with small probability p, otherwise *empty*. Subsequently, an empty site becomes active by contact with two or more active neighbors, and an occupied site becomes active if it has an active site within distance 2. We prove that the entire lattice becomes active with probability $\exp[\alpha(p)/p]$, where $\alpha(p)$ is between $-\pi^2/9 + c\sqrt{p}$ and $-\pi^2/9 + C\sqrt{p}(\log p^{-1})^3$. This corrects previous numerical predictions for the scaling of the correction term.

1 Introduction

Local bootstrap percolation is a 3-state cellular automaton on the square lattice $\mathbb{Z}^2$ defined as follows. At each time step $t = 0, 1, 2 \ldots$, each site of $\mathbb{Z}^2$ is either **empty** ($\circ$), **occupied** ($\bullet$), or **active** ($\star$). Let $p \in (0, 1)$. The *initial* configuration is given by a random element σ of $\{\circ, \bullet, \star\}^{\mathbb{Z}^2}$ under a probability measure $\mathbb{P}_p$ in which

$$\mathbb{P}_p\big[\sigma(0) = \star\big] = p, \quad \mathbb{P}_p\big[\sigma(0) = \circ\big] = 1 - p;$$
$$\mathbb{P}_p\big[\sigma(x) = \bullet\big] = p, \quad \mathbb{P}_p\big[\sigma(x) = \circ\big] = 1 - p, \quad (x \neq 0);$$

*Funded in part by NSF Grant DMS-0204376 and the Republic of Slovenia's Ministry of Science Program P1-285

†Funded in part by an NSERC (Canada) Discovery Grant, and by Microsoft Research

Key words: bootstrap percolation, cellular automaton, metastability, finite-size scaling, crossover

2000 Mathematics Subject Classifications: Primary 60K35; Secondary 82B43

and the states of different sites are independent. (Here $0 = (0,0) \in \mathbb{Z}^2$ is the origin). Subsequently, the configuration at time $t+1$ is obtained from that at t according to the following deterministic rules.

(L1) Each $\bullet$ becomes $\star$ if it has at least one $\star$ within ℓ^1-distance 2.
(L2) Each $\circ$ becomes $\star$ if it has at least two $\star$'s within ℓ^1-distance 1.
(L3) All other states remain unchanged.

We are interested in **indefinite growth**, i.e., the event that every site in $\mathbb{Z}^2$ eventually becomes active. The following is our main result.

Theorem 1 (growth probability)**.** *There exists constants $c, C, p_0 > 0$ such that for $p < p_0$,*

$$\exp \frac{-2\lambda + c\sqrt{p}}{p} \leq \mathbb{P}_p\big(\text{indefinite growth}\big) \leq \exp \frac{-2\lambda + C\sqrt{p}\,(\log p^{-1})^3}{p},$$

where $\lambda = \pi^2/18$.

Bootstrap percolation. As suggested by its name, the process we have introduced is a close relative of (standard) **bootstrap percolation**. This well-studied model may be described in our setting as follows. Initially, each site in the square $\{1, \ldots, L\}^2$ is independently active with probability p and empty otherwise, and all sites outside the square are empty. The configuration then evolves according to rules (L2,L3) ((L1) being irrelevant), and one is interested in the probability $I(L,p)$ that the entire square eventually becomes active.

Bootstrap percolation has been studied both rigorously and numerically [1, 2, 3, 17], both in its own right and as a tool to analyze other models [5, 6, 7, 8, 9]. An important property is that, asymptotically as $p \to 0$, it undergoes a sharp metastability transition as the parameter $p \log L$ crosses the threshold λ. More precisely, as proved in [12], $I(L,p)$ converges to 0 or 1 if respectively $p \log L < \lambda - \epsilon$ or $p \log L > \lambda + \epsilon$ (also see [12, 14] for similar results on related models). However, simulations for moderate values of p show surprising discrepancies with this rigorous result [6, 7, 10, 16], and our motivation in this article is to advance understanding of this latter phenomenon.

Local bootstrap percolation has been an implicit ingredient in many arguments involving bootstrap percolation, including those in [3, 10, 12], and a variant form appears explicitly in [6, 7] (see the discussion below). The proof of the main result from [12] already implies that for the local model, the quantity

$$-p \log \mathbb{P}_p(\text{indefinite growth})$$

converges to 2λ as $p \to 0$, and the main contribution of this paper is to identify the speed of this convergence up to logarithmic factors.

Heuristically, the dominant mechanism for active sites to take over space in bootstrap percolation is the presence of widely separated, and essentially independent, local bootstrap percolations. Indeed, for small p, bootstrap percolation immediately fixates (stops changing) on the overwhelming proportion of the lattice; the rare nuclei that facilitate growth hence encounter configurations of fixated active sites, which should be not very different from occupied sites in the local version. This suggests that $I(L,p)$ makes the transition from near 0 to near 1 as $L^2\, \mathbb{P}_p(\text{indefinite growth})$ changes from small to large, and thus when $2p \log L \approx -p \log \mathbb{P}_p(\text{indefinite growth})$. In this subject, claims not backed by rigorous arguments have a questionable track record; nevertheless, motivated by Theorem 1, we venture the following conjecture.

Conjecture 1. *In the standard bootstrap percolation model, let $L \to \infty$ and $p \to 0$ simultaneously. Then, with $\lambda = \pi^2/18$ as above and any $\epsilon > 0$, we have*

$$\textit{if} \quad L < \exp \frac{\lambda - p^{1/2-\epsilon}}{p} \qquad \textit{then} \quad I(L,p) \to 0.$$

The complementary bound, namely,

$$\text{if} \quad L > \exp \frac{\lambda - c\, p^{1/2}}{p} \qquad \text{then} \quad I(L,p) \to 1,$$

for a small enough $c > 0$, was proved in [10], so Conjecture 1 states that the power of p in the correction term is exactly $1/2$.

Rectangle process. It is natural to consider the following variant of the growth model defined by (L1–L3), in which we update states in a different order. A **rectangle** is a set of sites of the form $R = \{a, \ldots, c\} \times \{b, \ldots, d\} \subset \mathbb{Z}^2$. The **rectangle process** is a (random) sequence of rectangles $\rho_0 \subseteq \rho_1 \subseteq \rho_2 \subseteq \cdots$ defined in terms of the initial configuration σ as follows. If $\sigma(0) = \circ$ then we set $\rho_i = \emptyset$ for all i. If $\sigma(0) = \star$ then we set $\rho_0 = \{0\}$, and then proceed inductively as follows. If $\rho_i = \{a, \ldots, c\} \times \{b, \ldots, d\}$, then consider the configuration in which every site in ρ_i is active, every site outside the enlarged rectangle $\rho_i^+ := \{a-2,\dots,c+2\} \times \{b-2,\dots,d+2\}$ is empty, and all sites in $\rho_i^+ \setminus \rho_i$ have the same state as in σ. Apply the update rules (L1–L3) to this configuration until the configuration stops changing. It is readily seen that the resulting set of active sites is again a rectangle; let ρ_{i+1} be this set.

It is straightforward to check that the set of eventually active sites in the local bootstrap percolation model is identical to the limiting rectangle $\lim_{i\to\infty} \rho_i$, and in particular indefinite growth occurs if and only if the latter equals $\mathbb{Z}^2$.

The rectangle process can be described via a countable-state-space Markov chain, whose state represents the current rectangle together with information about which sites on its sides have been examined. In principle, this allows for a computational approach to estimating the probability of indefinite growth; as we do not presently pursue this, we will not give a detailed description of the chain.

Variant models. In addition to the standard model, our methods adapt to the following two variants. In the **modified local model**, we replace rule (L1) with:

(L1F) Each $\bullet$ becomes $\star$ if it has at least one $\star$ within ℓ^∞-distance 1.

In the **Froböse local model**, we replace (L1) with:

(L1M) Each $\bullet$ becomes $\star$ if it has at least one $\star$ within ℓ^1-distance 1.

These models may be regarded as local versions of **modified bootstrap percolation** [12, 13, 15] (in which an empty site becomes active if it has at least one active horizontal neighbor and at least one active vertical neighbour), and **Froböse bootstrap percolation** [9] (in which activation of a site requires a horizontal neighbor, a vertical neighbor, and the diagonal neighbor between them all to be active).

Theorem 2 (growth probability for variant models)**.** *Theorem 1 holds for the modified and Froböse local models, but with* $\lambda = \pi^2/6$. *For the Froböse model the upper bound can be improved to* $\exp[(-2\lambda + C\sqrt{p}\,(\log p^{-1})^2)/p]$.

The proof of Theorem 2 follows the same steps as that of Theorem 1 (with a few simplifications). We therefore omit the details and instead summarize the differences in Section 4.

Comparison with numerical results. For the modified and Froböse models, the resulting rectangle process, together with the associated Markov chain, is much simpler, which makes a computational approach more inviting. Indeed, in [7], using computer calculations together with heuristic arguments, the authors obtained the estimates

$$-p\,\log \mathbb{P}_p(\text{indefinite growth}) \approx 2\lambda - 6.22\,p^{0.333}$$

for the modified local model, and

$$-p\,\log \mathbb{P}_p(\text{indefinite growth}) \approx 2\lambda - 5.25\,p^{0.388}$$

for the Froböse local model, as $p \to 0$. These estimates seems to fit well down to $p \approx 0.01$, but Theorem 2 contradicts them in the limit, because the estimated powers 0.333 and 0.388 are less than 1/2.

In a similar vein, in [16], using interpolation between simulations and the rigorous result of [12], it was estimated for the standard (non-local) bootstrap percolation model, that the metastability transition occurs at

$$p \log L \approx \lambda - 0.51\,p^{0.2}.$$

This would again be inconsistent in the limit $p \to 0$ with our Conjecture 1.

These deceptively simple models thus present yet another puzzle of the type "why are we not able to see the asymptotic behavior in simulations?" That computer simulations can be so misleading [12, 17] is arguably the primary lesson learned from more than two decades of research into bootstrap percolation by mathematicians and physicists, and the present article is another contribution to this theme.

Outline of proof. We conclude the introduction with a brief outline of the rest of the paper. The bulk of the work, contained in Section 2, is in proving the upper bound in Theorem 1. We consider the rectangle process as the side length grows from $1/\sqrt{p}$ to $p^{-1}\log p^{-1}$. We sample this process in a sequence of coarse-grained steps, in which the side length increases by a factor of roughly $1+\sqrt{p}$ at each step. These step sizes are chosen to balance the entropy factors (related to the number of possibilities), and errors arising from the replacing true probabilities of growth with larger quantities. The latter are probabilities of "no double gap" conditions for growth from a smaller to a larger rectangle, as in [12]. One complication, contributing to the logarithmic error factor between our upper and lower bounds, is that corner sites between the two rectangles may allow growth in two directions.

The rectangle process can grow via rectangles of unequal dimensions; however, by the variational principle of [12], the most likely scenario is the most symmetric one, i.e., growth of squares. This yields the value of the constant λ.

The lower bound in Theorem 1 is a straightforward consequence of methods from [10], which we summarize in Section 3. The idea is to consider paths in the space of rectangle sizes that deviate from symmetric growth by approximately $1/\sqrt{p}$ at scale $1/p$ (and are therefore not prohibitively improbable). The increased entropy from such paths is enough to introduce an additional multiplicative factor of $\exp(c/\sqrt{p})$ to the bound on the growth probability.

2 Upper Bound

It will be convenient to assume throughout that the probability p is sufficiently small, e.g., $p < 0.1$ suffices. Then we denote

$$\begin{aligned} q = q(p) &:= -\log(1-p); \\ A = A(q) &:= \lceil 1/\sqrt{q} \rceil; \\ B = B(q) &:= \lfloor q^{-1} \log q^{-1} \rfloor. \end{aligned}$$

Note that $q = p + p^2/2 + p^3/3 + \cdots$ for small p. Throughout, we use $c, c_1, c_2, \ldots$ and $C, C_1, C_2, \ldots$ to denote positive (respectively small and large) absolute constants, which may in principle be explicitly computed. For a rectangle $R = \{a, \ldots, c\} \times \{b, \ldots, d\}$ we denote its **dimensions** $\dim(R) = (c - a + 1, d - b + 1)$.

We will bound the probability of indefinite growth by summing over possible trajectories for the rectangle process. However, the total number of trajectories is too large, so we need the following concept. Take a sequence of rectangles $R_1, \ldots, R_{n+1}$, where $n \geq 1$. Write $\dim(R_i) = (a_i, b_i)$ and $s_i := a_{i+1} - a_i$ and $t_i := b_{i+1} - b_i$. We call the sequence **good** if it has the following properties.

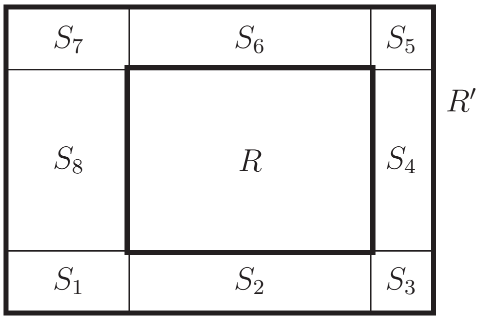

Figure 1: The rectangles $S_1, \dots, S_8$.

(i) $0 \in R_1 \subseteq R_2 \subseteq \cdots \subseteq R_{n+1}$.

(ii) $\min(a_1, b_1) \in [A, A+3]$.

(iii) $a_n + b_n \le B$.

(iv) $a_{n+1} + b_{n+1} > B$.

(v) For $i = 1, \dots, n$, either $s_i \ge a_i\sqrt{q}$ or $t_i \ge b_i\sqrt{q}$.

(vi) For $i = 1, \dots, n$, both $s_i < a_i\sqrt{q} + 4$ and $t_i < b_i\sqrt{q} + 4$.

We next define some useful events. The **columns** of a rectangle $R = \{a, \dots, c\} \times \{b, \dots, d\}$ are the sets $\{a\} \times \{b, \dots, d\}, \dots, \{c\} \times \{b, \dots, d\}$. We say that R has a **double gap in the columns** if two consecutive columns are entirely empty in the *initial* configuration σ. Double gaps in the **rows** are defined similarly. For a rectangle R, define the event

$$G(R) := \{R \text{ has no double gaps in the columns or rows}\}.$$

Moreover, we recall the following definition from [12]. For two rectangles $R \subseteq R'$, define subrectangles $S_1, \dots, S_8$ (some of which may be empty) according to Figure 1, so that R' is the disjoint union of $R, S_1, \dots, S_8$. Define $D(R, R')$ to be the event that each of the two rectangles $S_1 \cup S_8 \cup S_7$ and $S_3 \cup S_4 \cup S_5$ has no double gaps in the columns, and each of the two rectangles $S_1 \cup S_2 \cup S_3$ and $S_7 \cup S_6 \cup S_5$ has no double gaps in the rows. The idea is that this event is necessary for the growth to proceed from R to R'.

Finally, define the event

$$E := \bigcup \left\{ G(R) : \begin{array}{l} R \text{ contains } 0 \text{ and has one dimension in} \\ [B-A-10, B-A] \text{ and the other in } [1, A] \end{array} \right\}.$$

Lemma 3 (key inclusion)**.** *If indefinite growth occurs, then either E occurs, or else there exists a good sequence $R_1, \dots, R_{n+1}$ of rectangles such that*

$$G(R_1) \cap \bigcap_{i=1}^{n} D(R_i, R_{i+1})$$

occurs.

Proof. We start with the following observation: if the rectangle process ever encounters a given rectangle R, then $G(R)$ occurs. To see this, suppose on the contrary that R has two consecutive vacant columns, say. If these columns contain the origin then no growth occurs. Otherwise, so long as the growing rectangle includes no site above or below R, it contains no site in the two columns. This follows by induction on the steps of the rectangle process (no $\circ$ in the two columns can have two adjacent $\star$'s within R, while no $\bullet$ outside R can contribute to a site becoming $\star$ without first becoming $\star$ itself).

Now define the **good region** for dimensions of rectangles to be

$$T = \left\{(a, b) \in \mathbb{Z}^2 : a, b \geq A \text{ and } a + b \leq B\right\}$$

(see Figure 2). Note that in the sequence of dimensions $(\dim(\rho_j))_{j \geq 0}$ for the rectangle process, each coordinate increases by at most 4 at each step, while if indefinite growth occurs then at least one coordinate increases by at least 1 at every step, and both coordinates tend to ∞.

Therefore, assuming indefinite growth, if $(\dim(\rho_j))_{j \geq 0}$ never enters T, then E must occur, since the sequence must "escape" near a corner of the good region. On the other hand, if the sequence does enter T, then let R_1 be the first ρ_j such that $\dim(\rho_j) \in T$. Then property (ii) of a good sequence will be satisfied. Thereafter, define $R_2, \dots, R_{n+1}$ iteratively as follows. Assuming $\dim(R_i) = (a_i, b_i)$, let R_{i+1} be the first rectangle encountered by the rectangle process after R_i such that either $s_i \geq a_i \sqrt{q}$ or $t_i \geq b_i \sqrt{q}$ (where $(s_i, t_i) = \dim(R_{i+1}) - \dim(R_i)$). This ensures (v),(vi) are satisfied. Continue in this way, stopping at the first such rectangle, R_{n+1}, whose dimensions are outside T. Then (iii),(iv) are satisfied.

The rectangles thus constructed form a good sequence. From the above observations, $\cap_{i=1}^{n+1} G(R_i)$ occurs, and since $D(R, R') \subseteq G(R')$ the result follows. $\square$

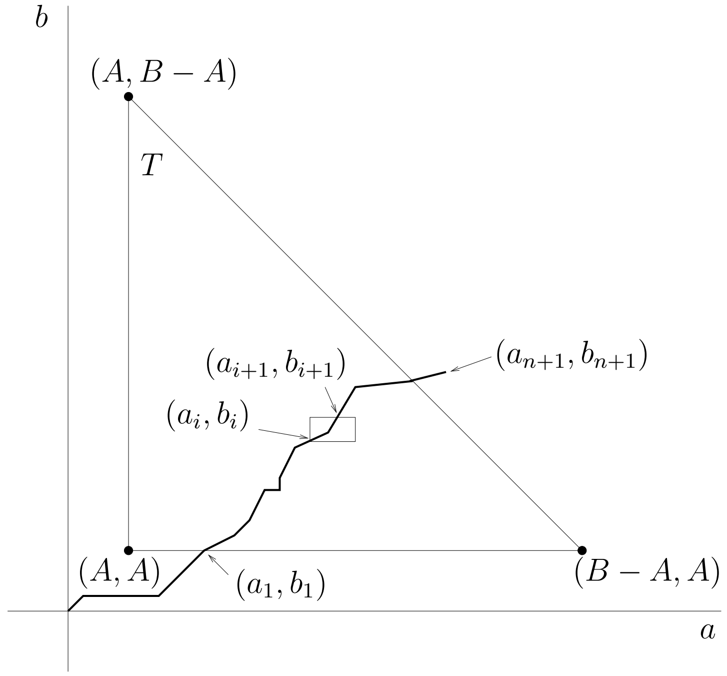

Figure 2: The sequence of dimensions (a, b) of the rectangle process (bold curve), as it passes through the good region T. The small rectangular window has dimensions $(a_i\sqrt{q}, b_i\sqrt{q})$.

Next we bound the probabilities of the events appearing in Lemma 3. We introduce the following functions from [12]:

$$\beta(u) = \frac{u + \sqrt{u(4-3u)}}{2} \quad \text{and} \quad g(z) = -\log \beta(1 - e^{-z}).$$

The threshold λ is determined by the integral (see [12], and also [14])

$$\int_0^\infty g(z)\, dz = \lambda = \frac{\pi^2}{18}. \tag{1}$$

Lemma 4 (double gaps)**.** *If R is a rectangle with* $\dim(R) = (a, b)$*,*

$$\mathbb{P}_p\big(R \text{ has no double gaps in the columns}\big) \le \exp\big[-(a-1)g(bq)\big].$$

For rows we have the same bound with a and b exchanged.

Proof. See [12, Lemma 8(i)]. ☐

The following is an enhanced version of [12, Proposition 21]. The reader may want to consult that proof for more details.

Lemma 5 (border event)**.** *For rectangles $R \subseteq R'$ of dimensions (a, b) and $(a + s, b + t)$,*

$$\begin{aligned}&\mathbb{P}_p\big[D(R, R')\big] \\ &\leq \exp\Big(-sg(bq) - tg(aq) + 2[g(bq) + g(aq)] + stq\, e^{2[g(bq)+g(aq)]}\Big).\end{aligned}$$

We remark that the second appearance of the constant 2 is crucial; the lemma would not hold with a smaller constant, while a larger constant would not suffice for the remainder of our argument, owing to the behaviour of g near zero (Lemma 10 below).

Proof of Lemma 5. We first claim that

$$\mathbb{P}_p\big[D(R, R')\big] \leq \sum_{k=0}^{st} \binom{st}{k} p^k (1-p)^{st-k} e^{-(s-2k-2)g(bq)} e^{-(t-2k-2)g(aq)}.$$

To prove this, we split the event according to the set of occupied sites in the set $S_1 \cup S_3 \cup S_5 \cup S_7$. If the number of such sites is k, they divide S_4 and S_8 into at most $k + 2$ contiguous vertical strips, each of which has no double gaps in the columns. A similar argument applies to rows in S_2 and S_6, and the relevant events are independent. Then use Lemma 4.

From the above, dropping the power of $(1 - p)$ and using the binomial expansion, we obtain

$$\mathbb{P}_p\big[D(R, R')\big] \leq e^{-sg(bq)-tg(aq)+2[g(bq)+g(aq)]} \left(1 + pe^{2[g(bq)+g(aq)]}\right)^{st}$$

Applying $1 + z \leq e^z$ and $p \leq q$ to the last factor finishes the proof. ☐

Lemma 6 (escape probability)**.** *For some absolute constant $c > 0$ and all $p < 0.1$,*

$$\mathbb{P}_p(E) \leq \exp\left[-cq^{-1}(\log q^{-1})^2\right].$$

Proof. By Lemma 4 and the definition of E,

$$\mathbb{P}_p(E) \leq 11A^2B \cdot \exp[(B - A - 11)g(Aq)].$$

The factor in front of the exponential comes from the number of possible choices of rectangles. Now use the definitions of A and B and the fact that

$$g(\epsilon) \sim \tfrac{1}{2} \log \epsilon^{-1} \quad \text{as } \epsilon \to 0. \qquad \square$$

We have provided all the probabilistic bounds we need, and we now proceed to analytic estimates of the expressions that appear. The following will be applied to the first two terms in the exponential in Lemma 3.

Lemma 7 (variational principle)**.** *Let A, B be any integers with $A > 4$ and $B > 2A$, and suppose $(a_i, b_i)_{i=1,\ldots,n+1}$ satisfy $a_0 = b_0 = A$, and $s_i := a_{i+1} - a_i \geq 0$ and $t_i := b_{i+1} - b_i \geq 0$, and properties (ii)–(iv) in the definition of a good sequence. For any $q > 0$ and any positive, smooth, convex, decreasing function $g : (0, \infty) \to (0, \infty)$,*

$$\sum_{i=0}^{n} \big[s_i g(b_i q) + t_i g(a_i q) \big] \geq \frac{2}{q} \int_{Aq}^{Bq} g - 2Bg(Bq/2).$$

Proof. We make use of the following technology from [12, Section 6]. If γ is a piecewise-linear path in the quadrant $(0, \infty)^2$, with each segment oriented in the non-negative direction of both co-ordinates, we define

$$w(\gamma) := \int_\gamma \Big(g(y)dx + g(x)dy \Big).$$

Let γ_1 be a piecewise-linear path with vertices $(a_0 q, b_0 q), \ldots, (a_{n+1} q, b_{n+1} q)$ (in that order). Since g is decreasing, the sum in the statement of the lemma is at least $w(\gamma_1)/q$ (compare [12, Proposition 16]).

Also let γ_2 be the straight line path from $(a_{n+1} q, b_{n+1} q)$ to (Bq, Bq), and let Δ be the straight path from (Aq, Aq) to (Bq, Bq). The variational principle in [12, Lemma 16] states that w is minimized by paths that follow the main diagonal, therefore

$$w(\gamma_1) + w(\gamma_2) \geq w(\Delta).$$

However, we have

$$w(\Delta) = 2 \int_{Aq}^{Bq} g.$$

On the other hand,

$$w(\gamma_2) \leq 2Bq \cdot g(Bq/2),$$

because $\min\{a_{n+1}, b_{n+1}\} > B/2$ and $(B - a_{n+1}) + (B - b_{n+1}) < 2B$. $\square$

Lemma 8 (integral)**.** *With q, A, B, g as defined previously, for $p < 0.1$ and an absolute constant C_1,*

$$\int_{Aq}^{Bq} g \geq \frac{\pi^2}{18} - C_1\sqrt{q}\log q^{-1}.$$

Proof. Use (1) together with the asymptotics

$$\int_0^{\epsilon} g \overset{\epsilon\to 0}{\sim} \frac{1}{2}\epsilon\log\epsilon^{-1} \quad \text{and} \quad \int_K^{\infty} g \overset{K\to\infty}{\sim} \frac{1}{2}e^{-2K}. \qquad \square$$

Lemma 9. *For some C_2 and all $p < 0.1$ we have $Bg(Bq/2) \leq C_2 \log q^{-1}$.*

Proof. As $g(K) \sim e^{-2K}$ as $K \to \infty$, and by the definition of B,

$$Bg(Bq/2) \leq \frac{1}{q}\log q^{-1} C' \cdot e^{-2\cdot\frac{2}{q}(q^{-1}\log q^{-1} - 1)}. \qquad \square$$

The next two lemmas will be used to bound the last two terms in the exponential in Lemma 3.

Lemma 10. *For every $a \leq B$ we have $e^{2g(aq)} \leq \frac{C_3}{aq}\log q^{-1}$.*

Proof. We have $e^{g(z)} = 1/\beta(1-e^{-z})$. Moreover, $\beta(1-e^{-z}) \sim z^{1/2}$ as $z \to 0$ and $\beta(1-e^{-z}) \to 1$ as $z \to \infty$. It follows that, for a large enough M,

$$\sup_{0<z\leq M} \frac{\sqrt{z}}{\beta(1-e^{-z})} \leq 2\sqrt{M}.$$

Therefore,

$$e^{2g(aq)} \leq \left(\frac{C'\sqrt{Bq}}{\sqrt{aq}}\right)^2. \qquad \square$$

Lemma 11 (summation bound)**.** *Let n and a_i, b_i $(i = 1, \dots, n+1)$ be positive integers and suppose that $s_i := a_{i+1} - a_i \geq 0$ and $t_i := b_{i+1} - b_i \geq 0$ for $i = 1, \dots, n$. Further, assume that (ii)–(vi) in the definition of a good sequence are satisfied. Then*

$$n \leq \frac{1}{\sqrt{q}}\log q^{-1} \quad \textit{and} \quad \sum_{i=1}^{n} \frac{s_i t_i}{a_i b_i} \leq C_4\sqrt{q}\log q^{-1}.$$

Proof. To bound n, use (v),(ii) and (iii) to get

$$(1+\sqrt{q})^n \le \frac{a_n b_n}{a_1 b_1} \le \frac{B^2}{A^2} \le \frac{1}{q}(\log q^{-1})^2.$$

Then use this to prove the second bound as follows. As $a_i \ge A$ and $b_i \ge A$ and $A\sqrt{q} \ge 1$, (vi) implies $s_i < 5a_i\sqrt{q}$ and $t_i < 5b_i\sqrt{q}$. Therefore

$$\sum_{i=1}^{n} \frac{s_i t_i}{a_i b_i} \le n \cdot 25q. \qquad \square$$

It will be convenient to bound the probability of $G(R_1)$ (associated with the entry of the rectangle process into the good region) in terms of a factor that can be easily combined with those obtained in Lemma 5, in order to use Lemma 7. The following lemma deals with this estimate.

Lemma 12 (entry term)**.** *Let $R_1, \ldots, R_{n+1}$ be a good sequence of rectangles and let $a_0 = b_0 := A$, with $s_0 := a_1 - a_0$ and $t_0 := b_1 - b_0$. Then, for some absolute constant C_5,*

$$\mathbb{P}_p\big[G(R_1)\big] \le \exp\big[-s_0 g(b_0 q) - t_0 g(a_0 q) + C_5 q^{-1/2} \log q^{-1}\big].$$

Proof. Without loss of generality suppose $a_1 \ge b_1$. By Lemma 4,

$$P[G(R_1)] \le e^{-(a_1-1)g(b_1 q)}.$$

Now $b_1 \le A+3$. We have $g'(z) \sim 1/(2\sqrt{z})$ as $z \to 0$, and g is convex. Therefore

$$0 \le g(Aq) - g(b_1 q) \le 3q \cdot g'(Aq) \le C_6\sqrt{q}.$$

Also $t_0 \le 3$, so

$$e^{t_0 g(a_0 q)} \le e^{3g(Aq)} \le C_7 q^{-3/2}.$$

Therefore,

$$\frac{\mathbb{P}_p[G(R_1)]}{\exp[-s_0 g(b_0 q) - t_0 g(a_0 q)]} \le C_7 q^{-3/2} \cdot e^{C_6 B\sqrt{q}}. \qquad \square$$

Lemma 13 (entropy)**.** *The number of good sequences of rectangles is at most*

$$\exp\left[\frac{C_8}{\sqrt{q}}(\log q^{-1})^2\right].$$

Proof. There are at most B^4 choices for R_1. For each subsequent rectangle there are at most $(B\sqrt{q}+4)^4$ choices, corresponding to the distance by which the rectangle grows in each of the four directions. The number of steps is $n+1$, which is bounded by Lemma 11. This gives the following upper bound:

$$B^4\,(B\sqrt{q}+4)^{4(q^{-1/2}\log q^{-1}+1)}. \qquad \square$$

We are now ready to prove the upper bound in the main theorem.

Proof of Theorem 1 (upper bound). Take $p < 0.1$. For a good sequence of rectangles, the events $G(R_1), D(R_i, R_{i+1})$ appearing in Lemma 3 are all independent, since they involve the initial states of disjoint sets of sites. Therefore,

$$\mathbb{P}_p\big(\text{indefinite growth}\big) \leq \mathbb{P}_p(E) + \sum_{R_1,\ldots,R_{n+1}} \mathbb{P}_p\big[G(R_1)\big] \prod_{i=1}^{n} \mathbb{P}_p\big[D(R_i, R_{i+1})\big], \tag{2}$$

where the sum is over all possible good sequences of rectangles. We now proceed to bound the various terms.

We start by using Lemmas 5 and 10 to bound $\mathbb{P}_p[D(R_i, R_{i+1})]$ above by

$$\begin{aligned}
&\exp\left[-s_i g(b_i q) - t_i g(a_i q)\right] \left(\frac{C_3}{Aq}\log q^{-1}\right)^2 \exp\left[s_i t_i q \frac{C_3}{a_i q}\log q^{-1} \frac{C_3}{b_i q}\log q^{-1}\right] \\
\leq &\exp\left[-s_i g(b_i q) - t_i g(a_i q)\right] \cdot \frac{C_9}{q}\left(\log q^{-1}\right)^2 \cdot \exp\left[\frac{C_9}{q}\left(\log q^{-1}\right)^2 \frac{s_i t_i}{a_i b_i}\right].
\end{aligned}$$

Using this and Lemma 12 we next bound $\mathbb{P}_p[G(R_1)] \prod_{i=1}^{n} \mathbb{P}_p[D(R_i, R_{i+1})]$ above by

$$\begin{aligned}
&\exp\left(-\sum_{i=0}^{n}\left[s_i g(b_i q) + t_i g(a_i q)\right]\right) \\
&\times \exp\left[\frac{C_5}{\sqrt{q}}\log q^{-1}\right] \cdot \left(\frac{C_9}{q}\left(\log q^{-1}\right)^2\right)^n \cdot \exp\left[\frac{C_9}{q}\left(\log q^{-1}\right)^2 \sum_{i=1}^{n}\frac{s_i t_i}{a_i b_i}\right].
\end{aligned}$$

By Lemma 11, this expression is at most

$$\exp\left(-\sum_{i=0}^{n}[s_i g(b_i q) + t_i g(a_i q)]\right) \cdot \exp\left[\frac{C_{10}}{\sqrt{q}}(\log q^{-1})^3\right].$$

Next, we use Lemmas 7, 8 and 9, to bound

$$\sum_{i=0}^{n} \left[s_i g(b_i q) + t_i g(a_i q) \right] \geq \frac{2\lambda}{q} - \frac{C_{11}}{\sqrt{q}} \log q^{-1}.$$

Finally, we substitute the last two bounds, together with Lemmas 13 and 6, into (2) to get

$$\begin{aligned} \mathbb{P}_p(\text{indefinite growth}) \quad &\leq \quad \exp\left[-cq^{-1}(\log q^{-1})^2\right] \\ + \quad \exp\left[\frac{C_8}{\sqrt{q}}(\log q^{-1})^2\right] &\cdot \exp\left[-\frac{2\lambda}{q} + \frac{C_{11}}{\sqrt{q}} \log q^{-1}\right] \cdot \exp\left[\frac{C_{10}}{\sqrt{q}}(\log q^{-1})^3\right]. \end{aligned}$$

Therefore

$$\mathbb{P}_p(\text{indefinite growth}) \leq \exp\left[-\frac{2\lambda}{q} + \frac{C'}{\sqrt{q}}(\log q^{-1})^3\right].$$

As a final step, use the asymptotics $q = p + \mathcal{O}(p^2)$ as $p \to 0$ to replace q with p. $\square$

3 Lower Bound

Proof of Theorem 1 (lower bound). This follows easily from the proof of Proposition 8 in [10]. In that article, it is proved that there is a certain event $\mathcal{F}$, satisfying

$$\mathbb{P}_p(\mathcal{F}) \geq \exp[-2\lambda/p + c/\sqrt{p}],$$

for small p, and defined in terms of the initial states in the quadrant $Q := \{0, 1, \ldots\}^2 \subset \mathbb{Z}^2$, on which the entire quadrant Q eventually becomes active in the bootstrap percolation model. On the same event (but replacing $\star$ with $\circ$ for sites $x \neq 0$), the entire quadrant becomes active in the local bootstrap percolation model also. This fact would not follow for an arbitrary event, but it holds for the particular event $\mathcal{F}$ because it is tailored to produce growth starting from the origin.

For the reader's convenience we briefly summarize the construction of the event $\mathcal{F}$, referring to [10] for the details. We consider two ways to grow from active square $[0, a]^2$ to the active square $[0, b]^2$. The first, symmetrical, way is that the top and right sides of the growing square always each have an occupied site within distance 2. The alternative, deviant, possibility is that vertical growth is temporarily halted by two adjacent vacant rows, and the square first elongates horizontally to the rectangle $[0,b]\times[0,a]$, when it can finally proceed to grow vertically until it reaches $[0,b]^2$. A key estimate, [10, Lemma 13], states roughly that, for a,b on the scale $1/p$, the quotient of probabilities of the second and the first event is at least $c_1\, p\, \exp\left[-2C\, p(b-a)^2\right]$. This suggests that one should keep $b-a$ of order $1/\sqrt{p}$ to incur a deviation cost $c_2 p$. Indeed, one can make about $m := c_3/\sqrt{p}$ deviant steps within these constraints, and a careful organization ensures that different choices of these steps give disjoint events. The resulting lower bound on $\mathbb{P}_p(\mathcal{F})$ is greater than $\exp\left[-2\lambda/p\right]$ by at least the factor

$$\binom{p^{-1}}{m}\left(1/\sqrt{p}\right)^m \cdot (c_2 p)^m \approx (c_2/c_3)^m,$$

which gives the required bound provided we choose $c_3 \ll c_2$.

Finally let $\mathcal{G}$ be the event that every semi-infinite line of sites, started at arbitrary site and moving in any direction parallel to one of the axes, contains at least one $\bullet$ initially. Clearly, $\mathbb{P}_p(\mathcal{G})=1$ and

$$\mathcal{F}\cap\mathcal{G}\subset\{\text{indefinite growth}\},$$

completing the proof. $\square$

4 Modified and Fröbose models

In this section we describe the proof of Theorem 2. The proof follows the same steps, and in fact a few simplifications are possible. We therefore summarize the differences.

For both the modified and Fröbose models, we replace the condition of "no double gaps" in the definitions of the events $G(R)$ and $D(R,R')$ with the condition that *no* columns (or rows) are initially entirely empty. The bound in Lemma 4 then becomes $\exp[-af(bq)]$, where

$$f(z) = -\log(1-e^{-z}).$$

The function f then replaces g throughout, and the threshold λ then arises as $\int_0^\infty f = \pi^2/6$. We also modify the definition of the rectangle process so that ρ^+ is ρ enlarged by only 1 in each direction.

For the modified model, the change to Lemma 4 allows the second 2 in the exponential in Lemma 5 to be replaced with 1. In fact this change is needed for the argument to go through, because of the different behavior of f near zero. Specifically, Lemma 10 holds with $e^{f(aq)}$ on the left side (and the same right side).

For the Froböse model we can make a further simplification to the argument, as horizontal and vertical progress now occur disjointly. To be more precise, $D(R, R')$ is now defined to be the event that the rectangle process makes a transition from R to R'. Then, referring to Figure 1, the two events:

$$S_1 \cup S_2 \cup S_3 \text{ and } S_5 \cup S_6 \cup S_7 \text{ have no empty rows, and}$$
$$S_1 \cup S_7 \cup S_8 \text{ and } S_3 \cup S_4 \cup S_5 \text{ have no empty columns,}$$

occur disjointly and are thus negatively correlated by the Van den Berg Kesten inequality (see e.g. [11]). This fact eliminates the last summand in the exponential in Lemma 5, makes Lemma 10 unnecessary and reduces the power of $\log p^{-1}$ to 2 in the final bound.

Open Problems

(i) Is a power of $\log p^{-1}$ in the upper bound of Theorem 1 really necessary?

(ii) In the modified and Froböse models, can the discrepancy between the asymptotic power 1/2 and the numerical estimates, which are closer to 1/3, be explained?

(iii) Is it possible to make the same arguments work if the distance in rule (L1) is altered to something larger than 2?

References

[1] J. Adler and U. Lev. Bootstrap percolation: Visualizations and applications. *Brazilian J. Phys.*, 33(3):641–644, 2003.

[2] J. Adler, D. Stauffer, and A. Aharony. Comparison of bootstrap percolation models. *J. Phys. A*, 22:L297–L301, 1989.

[3] M. Aizenman and J. L. Lebowitz. Metastability effects in bootstrap percolation. *J. Phys. A*, 21(19):3801–3813, 1988.

[4] J. Balogh and B. Bollobás. Sharp thresholds in bootstrap percolation. *Physica A*, 326(3):305–312, 2003.

[5] N. Cancrini, F. Martinelli, C. Roberto, and C. Toninelli. Kinetically constrained spin models. *Probab. Theory Related Fields* 140(3-4): 459–504, 2008.

[6] P. De Gregorio, A. Lawlor, P. Bradley, and K. A. Dawson. Exact solution of a jamming transition: closed equations for a bootstrap percolation problem. *Proc. Natl. Acad. Sci. USA*, 102(16):5669–5673 (electronic), 2005.

[7] P. De Gregorio, A. Lawlor, and K. A. Dawson. New approach to study mobility in the vicinity of dynamical arrest; exact application to a kinetically constrained model. *Europhys. Lett.*, 74(2):287–293, 2006.

[8] L. R. Fontes, R. H. Schonmann, and V. Sidoravicius. Stretched exponential fixation in stochastic Ising models at zero temperature. *Comm. Math. Phys.*, 228(3):495–518, 2002.

[9] K. Froböse. Finite-size effects in a cellular automaton for diffusion. *J. Statist. Phys.*, 55(5-6):1285–1292, 1989.

[10] J. Gravner, and A. E. Holroyd. Slow convergence in bootstrap percolation. *Ann. Appl. Prob.*, 18(3):909–928, 2008.

[11] G. R. Grimmett. *Percolation.* Springer-Verlag, second edition, 1999.

[12] A. E. Holroyd. Sharp metastability threshold for two-dimensional bootstrap percolation. *Probab. Theory Related Fields*, 125(2):195–224, 2003.

[13] A. E. Holroyd. The metastability threshold for modified bootstrap percolation in d dimensions. *Electron. J. Probab.*, 11:no. 17, 418–433 (electronic), 2006.

[14] A. E. Holroyd, T. M. Liggett, and D. Romik. Integrals, partitions, and cellular automata. *Trans. Amer. Math. Soc.*, 356(8):3349–3368, 2004.

[15] R. H. Schonmann, On the behavior of some cellular automata related to bootstrap percolation. *Ann. Probab.*, 20(1): 174–193, 1992.

[16] D. Stauffer, Work described in [1], 2003.

[17] A. C. D. van Enter. Proof of Straley's argument for bootstrap percolation. *J. Statist. Phys.*, 48(3-4):943–945, 1987.

Alexander E. Holroyd: holroyd@math.ubc.ca
Department of Mathematics, University of British Columbia,
121-1984 Mathematics Rd, Vancouver, BC V6T 1Z2, Canada

Janko Gravner: gravner@math.ucdavis.edu
Mathematics Department, University of California,
Davis, CA 95616, USA