Stable Poisson Graphs in One Dimension

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April 2011

Abstract

Let each point of a homogeneous Poisson process on $\mathbb{R}$ independently be equipped with a random number of stubs (half-edges) according to a given probability distribution μ on the positive integers. We consider schemes based on Gale-Shapley stable marriage for perfectly matching the stubs to obtain a simple graph with degree distribution μ. We prove results on the existence of an infinite component and on the length of the edges, with focus on the case $\mu(\{2\}) = 1$. In this case, for the random direction stable matching scheme introduced by Deijfen and Meester we prove that there is no infinite component, while for the stable matching of Deijfen, Häggström and Holroyd we prove that existence of an infinite component follows from a certain statement involving a *finite* interval, which is overwhelmingly supported by simulation evidence.

1   Introduction

Let $\mathcal{P}$ be a homogeneous Poisson process with intensity 1 on $\mathbb{R}^d$ and μ a probability measure on the strictly positive integers. We shall study translation-invariant simple random graphs whose vertices are the points of $\mathcal{P}$ and where,

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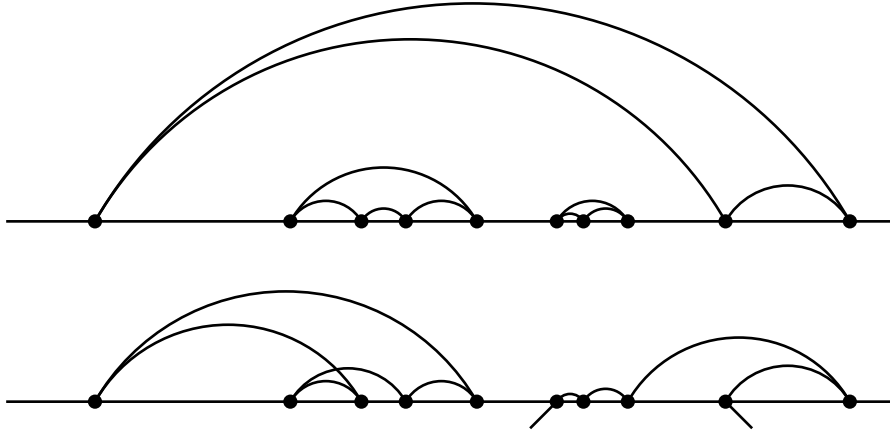
Figure 1: The stable multi-matching (top) and the random direction stable multi-matching, for 10 vertices on a finite interval, with 2 stubs per vertex.

conditional on $\mathcal{P}$, the degrees of the vertices are i.i.d. with law μ. Previously, Deijfen [2] has studied achievable moment properties for the edges, and Deijfen, Häggström and Holroyd [3] have studied the question of whether the graph contains a component with infinitely many vertices. In the latter work a particular matching scheme, called the stable multi-matching, was introduced, leading to a number of challenging open questions. Here we restrict to $d = 1$ and the focus is on the case $\mu(\{2\}) = 1$, one of the simplest cases for which the question of existence of an infinite component is non-trivial. For the stable multi-matching and a variant of it with prescribed random stub directions, we prove results on the component structure and on the length of the edges. Figure 1 shows schematic pictures of the two matchings, which are described below.

First we formally describe the objects that we will work with. Write $[\mathcal{P}] := \{x \in \mathbb{R} : \mathcal{P}(\{x\}) > 0\}$ for the support, or point-set, of $\mathcal{P}$. Let ξ be a random integer-valued measure on $\mathbb{R}$ with the same support as $\mathcal{P}$, and which, conditional on $\mathcal{P}$, assigns i.i.d. values with law μ to the elements of $[\mathcal{P}]$. The pair $(\mathcal{P}, \xi)$ is a marked point process with positive integer-valued marks. For $x \in [\mathcal{P}]$ we write D_x for $\xi(\{x\})$ and interpret this as the number of stubs at vertex x.

A **matching scheme** for a marked process $(\mathcal{P}, \xi)$ is a point process $\mathcal{M}$ on the space of unordered pairs of points in $\mathbb{R}$, with the property that almost surely for every pair $(x, y) \in [\mathcal{M}]$ we have $x, y \in [\mathcal{P}]$, and such that

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in the graph $G = G(\mathcal{P}, \mathcal{M})$ with vertex set $[\mathcal{P}]$ and edge set $[\mathcal{M}]$, each vertex x has degree D_x. Our primary interest is in the connected components of G. The matching schemes under consideration will always be **simple**, meaning that G has almost surely no self-loops and no multiple edges, and **translation-invariant**, meaning that $\mathcal{M}$ is invariant in law under the action of all translations of $\mathbb{R}$. We say that a translation-invariant matching is a **factor** if it is a deterministic function of the Poisson process $\mathcal{P}$ and the mark process ξ, that is, if it does not involve any additional randomness. We write $\mathbb{P}$ and $\mathbb{E}$ for probability and expectation on the probability space supporting the random triplet $(\mathcal{P}, \xi, \mathcal{M})$.

Let $(\mathcal{P}^*, \xi^*, \mathcal{M}^*)$ be the Palm versions of $(\mathcal{P}, \xi, \mathcal{M})$ with respect to $\mathcal{P}$ and write $\mathbb{P}^*$ and $\mathbb{E}^*$ for the associated probability law and expectation operator. Informally speaking, $\mathbb{P}^*$ describes the conditional law of $(\mathcal{P}, \xi, \mathcal{M})$ given that there is a point at the origin, with the mark process and the matching scheme taken as stationary background; see e.g. [9, Chapter 11] for more details. Since $\mathcal{P}$ is a Possion process, we have $[\mathcal{P}^*] \overset{d}{=} [\mathcal{P}] \cup \{0\}$.

We now define the two matching schemes that will be analyzed in the paper.

Stable multi-matching

The concept of stable matching was introduced by Gale and Shapley [5]. It has been studied in [7] and [8] in the context of spatial point processes (with $\mu(\{1\}) = 1$ in our notation). A natural generalization to other degree distributions μ was introduced in [3] and is referred to as the stable multi-matching. Formally, a matching scheme $\mathcal{M}$ is said to be a **stable multi-matching** if a.s., for any two distinct points $x, y \in [\mathcal{P}]$, either they are linked by an edge or at least one of them has no incident edges longer than $|x - y|$. Here and throughout, distance and edge length refer to the Euclidean norm $|\cdot|$ on $\mathbb{R}$.

We will restrict our attention to the case when $\mathcal{P}$ is a Poisson process. For this case, it was proved in [3, Proposition 2.2] that there is an a.s. unique stable multi-matching, which moreover can be constructed by the following iterative procedure. First connect all mutually closest pairs of points in $[\mathcal{P}]$ and remove one stub from each of the points just matched. Then call two points compatible if they do not already have an edge between them and if both of them have at least one stub left. Connect all mutually closest compatible pairs and remove one stub from each of the points just matched. Repeat indefinitely. See [3, Propostion 2.2].

Random direction stable multi-matching

We introduce a variant of stable multi-matching where the directions of the edges are prescribed independently of the Poisson process. As described above, the process ξ assigns a mark D_x to each point $x \in [\mathcal{P}]$. Let ψ be a second mark process which, conditionally on $\mathcal{P}$ and ξ, assigns an integer $R_x \sim \text{Binomial}(D_x, 1/2)$ independently to each point $x \in [\mathcal{P}]$. We think of R_x as the number of stubs incident with x that are to be matched to the right of x. If $x < y$, and (x, y) is an edge of a matching scheme $\mathcal{M}$, we call (x, y) a right-edge of x, and a left-edge of y. A matching scheme $\mathcal{M}$ is now said to be a **random direction stable multi-matching** if each point $x \in [\mathcal{P}]$ has exactly R_x incident right-edges and if a.s., for any two distinct points $x, y \in [\mathcal{P}]$ with $x < y$, either they are linked by an edge, or x has no incident right-edges longer than $|x - y|$, or y has no incident left-edges longer than $|x - y|$.

Let each point $x \in [\mathcal{P}]$ be equipped with R_x stubs pointed to the right and $L_x := D_x - R_x$ stubs pointed to the left, and consider the following iterative procedure for matching right-stubs to left-stubs. First consider all pairs of consecutive points in $[\mathcal{P}]$. Create an edge between every such pair $x < y$ such that x has at least one right-stub and y has at least one left-stub, and remove the corresponding stubs. Then consider pairs of points in $[\mathcal{P}]$ with precisely one point in $[\mathcal{P}]$ in between them. Create an edge between every such pair of points $x < y$ such that x has at least one right-stub and y has at least one left-stub left, and remove the corresponding stubs. Continue indefinitely, with pairs of points separated by an increasing number of points. This procedure has previously been studied in [4]. We show in Section 2 that it leads to the unique stable multi-matching subject to the prescribed (random) directions for the edges.

1.1 Results

In this section we collect the main results. The proofs are then given in Section 3. The first result concerns uniqueness of the infinite component.

Proposition 1.1. *For a Poisson process on $\mathbb{R}$ and any degree distribution, in the stable multi-matching and the random direction stable multi-matching, there is at most one infinite component.*

The next result asserts that, in the case $\mu(\{2\}) = 1$ of two stubs per

vertex, the random direction stable multi-matching has no infinite components. For other degree distributions the existence of an infinite component remains an open question. Part (b) of the theorem however provides some information on the edge length. See [4, Theorem 4.1] and [7, Theorem 2] for related results.

For $x \in [\mathcal{P}]$, let X_x denote the average length of all edges incident to x, and write $X = X_0$ for the value at the origin in the Palm version of the process.

Theorem 1.2. *For a Poisson process on $\mathbb{R}$, consider the random direction stable multi-matching.*

(i) *For $\mu(\{2\}) = 1$, almost surely there is no infinite component.*

(ii) *For any degree distribution with bounded support, we have $\mathbb{E}^*[X^{1/2}] = \infty$.*

Turning to the stable multi-matching, it was proved in [3, Theorem 1.2(b)] that there is no infinite component when the only possible values for the degrees are 1 and 2, with a strictly positive probability of degree 1. In $d \geq 2$ it was also proved that there is an integer $k = k(d)$ such that if all vertices almost surely have degree at least k, then there is almost surely an infinite component, [3, Theorem 1.2(a)]. Note that, by ergodicity, the event that there exists an infinite component has probability 0 or 1 for any degree distribution. The following result relates the existence of an infinite component for the case $\mu(\{1,2\}) = 1$ in $d = 1$ to a certain property concerned with the lengths of the edges. Let M_x denote the length of the longest edge incident to $x \in [\mathcal{P}]$, say that x **desires** a site $y \in \mathbb{R}$ if $|y - x| < M_x$ and write N for the number of points in $[\mathcal{P}]$ that desire the origin.

Theorem 1.3. *For a Poisson process on $\mathbb{R}$, consider the stable multi-matching.*

(i) *For any degree distribution, if there is no infinite component, then $N = \infty$ almost surely.*

(ii) *If $\mu(\{1,2\}) = 1$ and there is an infinite component, then $N < \infty$ almost surely.*

For degree distributions with $\mu(\{1,2\}) = 1$, existence of an infinite component in the stable multi-matching is hence equivalent to $N < \infty$. On the other hand, N is related to edge lengths, as follows. Write $M = M_0$.

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Lemma 1.4. *For any translation-invariant matching scheme, we have that* $\mathbb{E}^*[M] < \infty$ *if and only if* $\mathbb{E}[N] < \infty$.

In view of this relation, $E^*[M] < \infty$ would imply that $N < \infty$, and thereby establish the existence of an infinite component for $\mu(\{2\}) = 1$ in the stable multi-matching. However, the best result we have in this direction is the following, which applies in any dimension $d \geq 1$ (the stable multi-matching is defined analogously in all dimensions; see [3]).

Proposition 1.5. *For a Poisson process of intensity* 1 *on* $\mathbb{R}^d$, *and any degree distribution with bounded support, in the stable multi-matching we have* $\mathbb{P}^*(M > t) \leq ct^{-d}$ *for some* $c \in (0, \infty)$ *(depending only on* d *and the bound on degree).*

A "statistical proof" of percolation. It is not rigorously known whether the stable multi-matching with $\mu(\{2\}) = 1$ has an infinite component in $d = 1$. However, in Section 4 we will present compelling evidence that this is indeed the case. Specifically, we will define a certain event G_L *in terms of a Poisson process on the bounded interval* $[0, L]$. We will prove rigorously that for any $L > 0$,

$$\mathbb{P}(G_L) > 0.968 \quad \text{implies existence of an infinite component.}$$

On the other hand, since G_L is defined in terms of a bounded interval, its probability can be estimated by Monte-Carlo simulation. Such simulations provide overwhelming statistical evidence that

$$\mathbb{P}(G_{13000}) > 0.968.$$

The random direction stable multi-matching and the stable multi-matching are hence qualitatively different: when the directions of the stubs are prescribed randomly, there is no infinite component, while when the directions are prescribed by the positions of the Poisson points (as in the stable multi-matching) there is an infinite component. Figure 2 shows simulation pictures of the random direction stable multi-matching and the stable multi-matching, respectively, with $\mu(\{2\}) = 1$ in $d = 1$.

Say that an edge $(x, y) \in [\mathcal{M}]$ **crosses** a site $z \in \mathbb{R}$ if $x < z < y$. Our last result is the following.
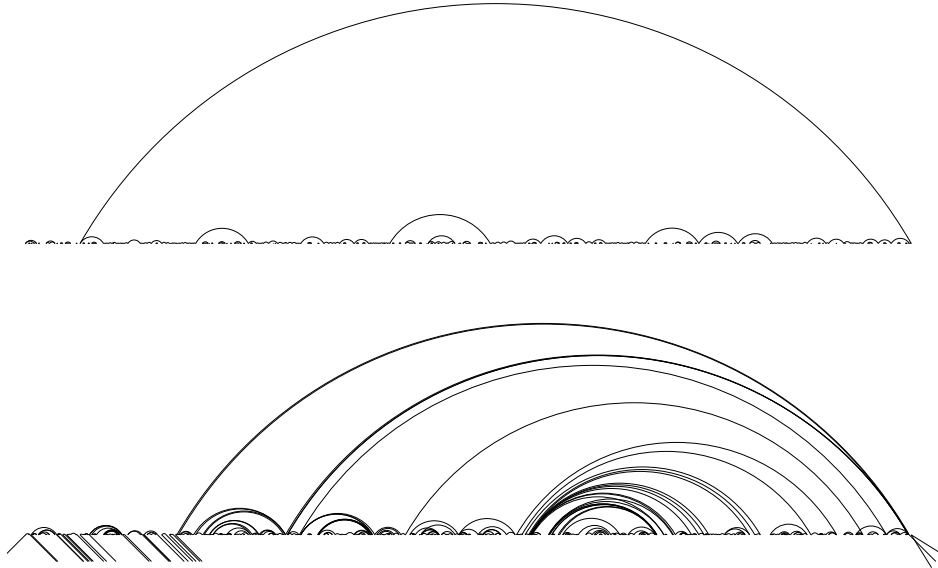
Figure 2: The stable multi-matching (top) and the random direction stable multi-matching, with 2 stubs per vertex, for 500 uniformly random points on an interval.

Proposition 1.6. *For a Poisson process on $\mathbb{R}$, and any degree distribution μ whose support includes some odd integer, for any factor matching scheme, the number of edges that cross the origin is infinite.*

If the number of edges that cross the origin if infinite, then clearly also $N = \infty$. Hence, appealing to Lemma 1.4, Proposition 1.6 implies that $\mathbb{E}^*[M] = \infty$ in any factor matching scheme for degree distributions whose support contains an odd integer. Since the stable multi-matching is a factor, combining Proposition 1.6 with Theorem 1.3(b) gives an alternative proof (in $d = 1$) of the result of [3] that the stable multi-matching does not percolate when the only possible values for the degrees are 1 and 2 and the probability of degree 1 is strictly positive. For degree distributions with support on larger values this approach is inconclusive, since Theorem 1.3(b) does not apply.

The rest of the paper is organized as follows. In Section 2, a few preliminary results are collected. The above results are then proved in Section 3. In Section 4, existence of a infinite component in the degree 2 case in $d = 1$ is shown to follow from the assertion that a certain finite event has large

enough probability, which is convincingly supported by Monte-Carlo simulation. Section 5 contains examples demonstrating that there is no general relation between the edge length and the existence of an infinite component valid for any matching scheme, and that for point processes other than the Poisson process, both percolation and non-percolation are possible for the stable multi-matching in the degree 2 case. Finally, in Section 6 some directions for further work are presented. For background on the problem we refer to [3, Section 2.1] and [2, Section 1].

2   Preliminaries

We first show that the iterative procedure described for the stable multi-matching with random directions leads to the unique stable multi-matching with the prescribed directions for the edges.

Proposition 2.1. *Let* $(\mathcal{P}, \xi, \psi)$ *be a doubly marked Poisson process. Almost surely, the iterative multi-matching procedure described in the introduction exhausts the set of stubs, and the limiting graph (after an infinite number of iterations) is a random direction stable multi-matching. No other such matching scheme exists.*

Proof. Let $\mathcal{P}'_r$ (respectively $\mathcal{P}'_l$) be the process of points with at least one unmatched right-stub (left-stub) on them after the above matching procedure is completed. By symmetry $\mathcal{P}'_r$ and $\mathcal{P}'_l$ have the same intensity and they are both ergodic point processes. Hence either both have a.s. infinitely many points or both have a.s. no points. The first option however would produce a contradiction, since the iterative procedure could then be applied to the remaining configuration of stubs giving rise to edges that would have been created already in the original procedure.

That the resulting multi-matching is stable subject to the prescribed (random) directions follows from the definition: an unstable pair of points – that is, a pair x and y with $x < y$ with no edge between them and where x (y) has an edge connected to the right (left) of y (x) – would have had an edge created between them at some stage of the matching procedure. That it is the unique matching with this property follows by induction over the stages in the algorithm to show that each edge that is present in the resulting configuration must be present in any stable matching of the stubs with the prescribed directions. □

We proceed by formulating a version of the mass transport principle suitable for our needs. For background, see [1]. A **mass transport** is a random measure T on $(\mathbb{R}^d)^2$ that is invariant in law under translations of $\mathbb{R}^d$. For Borel sets $A, B \subset \mathbb{R}^d$, we interpret $T(A, B)$ as the amount of mass transported from A to B. Write Q for the unit cube $[0, 1)^d$.

Lemma 2.2 (Mass Transport Principle). *Let T be a mass transport. Then*

$$\mathbb{E}\, T(Q, \mathbb{R}^d) = \mathbb{E}\, T(\mathbb{R}^d, Q)\,.$$

Proof.

$$\mathbb{E}\, T(Q, \mathbb{R}^d) = \sum_{z \in \mathbb{Z}^d} \mathbb{E}\, T(Q, Q + z) = \sum_{z \in \mathbb{Z}^d} \mathbb{E}\, T(Q - z, Q) = \mathbb{E}\, T(\mathbb{R}^d, Q)\,. \quad \square$$

Lemma 1.4 is now easily established using the mass transport principle.

Proof of Lemma 1.4. Consider the mass transport in which each point $x \in [\mathcal{P}]$ sends out mass $2M_x$, and distributes it uniformly to the interval $(x - M_x, x + M_x)$. The expected mass sent out from the unit interval $[0, 1)$ equals $2\,\mathbb{E}^*[M]$. On the other hand, the mass received by the unit interval is $\int_0^1 N_x\, dx$, where N_x denotes the number of points that desire $x \in \mathbb{R}$. Hence the expected mass received by the unit interval is $\mathbb{E}N$. The result hence follows from the mass transport principle. $\qquad\square$

Next we observe that an infinite component in a translation-invariant matching scheme must be unbounded both to the right and to the left, that is, for any $r \in \mathbb{R}^+$ it must contain points both to the right of r and to the left of $-r$.

Lemma 2.3. *A translation-invariant matching scheme almost surely cannot have an infinite component that is unbounded in only one direction.*

Proof. Assume that there is a matching scheme that with positive probability gives rise to an infinite component that is unbounded in only one direction, say to the left, and consider the mass transport in which each vertex in such an infinite component sends mass 1 to the rightmost point in the component. With positive probability such a rightmost point is located in the unit interval, which then receives infinite mass. The expected mass sent out from the unit interval is however bounded by 1, so we have a contradiction with the mass transport principle. $\qquad\square$

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Finally, the following result will be of use in proving Theorem 1.3(b).

Lemma 2.4. *Let Γ be a translation-invariant simple point process of finite intensity on $\mathbb{R}$. For $x \in [\Gamma]$, write Z_x for the maximum of the distances from x to the nearest point of $[\Gamma]$ on the left and the nearest point on the right. The number of points $x \in [\Gamma]$ with $Z_x > |x|$ is finite almost surely.*

Proof. Consider the mass transport in which an interval sends out mass equal to its length, and the mass sent out by an interval (x, y) between consecutive points $x < y$ of $[\Gamma]$ is distributed uniformly to the interval $(x-(y-x), y+(y-x))$. If there were infinitely many points $x \in [\Gamma]$ with $Z_x > |x|$, then the unit interval would receive infinite mass, which conflicts with the mass transport principle, since the mass sent out from the unit interval equals 1. $\qquad\square$

3   Proofs

We now proceed to prove the results in Section 1.1, starting with the uniqueness of an infinite component. We say that two edges (a, b) and (c, d) in $[\mathcal{M}]$ **cross** each other if $a < c < b < d$.

Proof of Proposition 1.1. Observe that both matching schemes have the property that,

$$\text{if two edges } (a, b) \text{ and } (c, d) \text{ cross each other, then}$$
$$\text{the edge } (c, b) \text{ must also be present in the matching.} \tag{1}$$

– this follows from the definitions of the matching schemes. Two distinct components hence cannot have crossing edges. However, by Lemma 2.3, any infinite component must be unbounded in both directions. Hence two distinct infinite components would necessarily have crossing edges. $\qquad\square$

Proof of Theorem 1.2 (i). Recall that L_x (respectively R_x) is the number of edges incident with $x \in [\mathcal{P}]$ that are connected to the left (right) of x. Let $\mu(\{2\}) = 1$. We call x a **bird** if $L_x = R_x = 1$, a **left-beak** if $L_x = 2$ and a **right-beak** if $R_x = 2$ (see Figure 1). Let $(\cdots <)x_1 < \cdots < x_k(< \cdots) \in [\mathcal{P}]$ be the ordered vertices of some (finite or infinite) component of the stable multi-matching (recall by Lemma 2.3 that a component is either finite or unbounded in both directions). Clearly, if the component is finite, then its leftmost point is a right-beak and its rightmost point is a left-beak. We

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claim that if x_i is a right-beak, and not one of these extreme points of the component, then x_{i+1} is a left-beak. To check this, let the two edges from x_i have their other endpoints at x_j and x_k, where $x_i < x_j < x_k$. We claim that the other neighbour of x_j must lie left of x_i. To see this, follow the path formed by the cluster starting with the edge (x_i, x_j) - eventually we must leave the interval $[x_i, x_k]$, since the cluster contains points to the left of x_i. When we do so, it is via an edge that crosses (x_i, x_k). Unless it is the first edge encountered after x_j, this entails a violation of (1). Thus x_j is a left beak. Now there cannot be any further vertices of the cluster in the interval $[x_i, x_j]$, since such a vertex would have an incident edge crossing (x_i, x_j), again contradicting (1).

Therefore, the non-extreme vertices of a component consist of birds together with consecutive right-beak/left-beak pairs. Note that between the points of a component there may be points belonging to other components, but since two components cannot have crossing edges, any other such component must lie in a single interval (x_i, x_{i+1}).

Now consider the function $F : \mathbb{R} \to \mathbb{Z}$ defined by $F(0) = 0$, and

$$F(y) - F(x) = \sum_{t \in [\mathcal{P}] \cap [x,y)} (L_t - R_t), \quad x < y.$$

Thus, F takes a up-step (of size 2) at each left-beak and a down-step at each right-beak. Hence it is a continuous-time simple symmetric random walk on the even integers. On the other hand, by the observations above concerning components, if there is an infinite component, then F is bounded above a.s. by some (random) constant, which is impossible. □

Proof of Theorem 1.2 (ii). We use a variant of an argument from [7, Theorem 2(b)]. For $A \subset \mathbb{R}$, write $R(A)$ for the total number of right-stubs at points $x \in [\mathcal{P}] \cap A$, that is, $R(A) = \sum_{x \in [\mathcal{P}] \cap A} R_x$, and define $L(A)$ analogously as the total number of left-stubs in A. Furthermore, for $A, B \subset \mathbb{R}$, let $R(A \to B)$ denote the number of right-stubs in A that are matched with left-stubs in B, and $D(A \leftrightarrow B)$ the total number of edges connecting points in $[\mathcal{P}] \cap A$ to points in $[\mathcal{P}] \cap B$. Write k for the supremum of the support of

μ. For $t > 0$, we have

$$
\begin{aligned}
\mathbb{E}R\big([0,2t] \to [0,2t]^c\big) &= \frac{1}{2}\,\mathbb{E}D\big([0,2t] \leftrightarrow [0,2t]^c\big) \\
&\leq \frac{k}{2}\int_0^{2t} \mathbb{P}^*\big(X > x \wedge (2t - x)\big)\,dx \\
&= k \cdot \mathbb{E}^*[X \wedge t].
\end{aligned}
$$

Furthermore, since μ has bounded support, we can use the central limit theorem to get that

$$
\mathbb{E}R\big([0,2t] \to [0,2t]^c\big) \geq \mathbb{E}\left[\big(R[0,2t] - L[0,2t]\big)^+\right] \sim ct^{1/2}
$$

as $t \to \infty$ for some $c > 0$. Hence $t^{-1/2}\mathbb{E}^*[X \wedge t] \geq c'$ for sufficiently large t and some $c' > 0$. On the other hand, if $\mathbb{E}^*[X^{1/2}] < \infty$, then $t^{-1/2}\mathbb{E}^*[X \wedge t] \to 0$ as $t \to \infty$ by the dominated convergence theorem, a contradiction. $\qquad\square$

Proof of Theorem 1.3 (i). Let

$$
H = \{x \in [\mathcal{P}] : M_x > |x| - 1\},
$$

that is, H is the set of vertices that desire some point in the unit interval $(-1, 1)$. Write $\widetilde{N}$ for the cardinality of H. We will show that $\widetilde{N} = \infty$ a.s. By symmetry this implies that with positive probability infinitely many vertices in $(1, \infty)$ desire 1. However, on the latter event, for any $a > 1$, infinitely many vertices in (a, ∞) desire a, so by ergodicity it follows that $N = \infty$ a.s.

First we show that $\mathbb{P}(\widetilde{N} = 0) = 0$. Assume for contradiction that $\mathbb{P}(\widetilde{N} = 0) > 0$. For a configuration $(\mathcal{P}, \xi)$ with $\tilde{N} = 0$, consider a modified configuration where a vertex is added uniformly at random in $[0, 1]$ independently of $\mathcal{P}$. If follows from [7, Lemma 18] and a straightforward modification of [7, Lemma 16] that all stubs at this vertex would be unmatched in the stable multi-matching, which contradicts [3, Proposition 2.2].

Now assume that all components are finite a.s., and suppose that for a contradiction that $\mathbb{P}(\widetilde{N} < \infty) > 0$. For a configuration $(\mathcal{P}, \xi)$ with $\widetilde{N} < \infty$, consider a modified configuration where the vertices in H are removed, along with all vertices in their components. The number of vertices that are removed is almost surely finite. But in this configuration, we have $\widetilde{N} = 0$, which is a contradiction to a straightforward modification of [7, Lemma 18]. $\qquad\square$

Proof of Theorem 1.3 (ii). First note that, when the only possible values for the degrees are 1 and 2, the stable multi-matching cannot contain any crossing edges. If $a < c < b < d$ and the edges (a, b) and (c, d) are present, then, as pointed out in the proof of Proposition 1.1, the edge (c, b) must also be present in the matching. But if $b - a > d - b$, then b and d desire each other, and are hence connected by an edge, so b has degree at least 3. Similarly, if $b - a < d - b$ then c has degree at least 3.

Lemma 2.3 and the fact that edges do not cross imply that an infinite component must consist of a set of degree-2 vertices, unbounded in both directions, with an edge between each consecutive pair. It follows from Lemma 2.4 that the number of vertices in this infinite component that desire the origin is finite almost surely.

As for the finite components, each must be contained in a single interval defined by an edge of the infinite component (since there are no crossing edges). Note also (although this observation will not be needed) that a component of size k must consist of vertices $x_1 < \cdots < x_k$ with edges (x_i, x_{i+1}) for all $i = 1, \ldots, k-1$ together with the edge (x_1, x_k).

Now let I_0 denote the interval defined by the edge in the infinite path that crosses the origin. This interval is finite and hence contains almost surely finitely many points of $[\mathcal{P}]$ in finite components. These points might desire the origin. A vertex $x > 0$ (respectively, $x < 0$) in a finite component outside this interval however cannot desire the origin: if it did, it would also desire the left-most (right-most) end-point of the interval I_x defined analogously to I_0. But this vertex also desires x, which means that there would be an edge between them.

We conclude that $N < \infty$ almost surely, as desired. $\qquad\square$

Proof of Proposition 1.5. Say that a point $x \in [\mathcal{P}]$ is t-**bad** if $M_x > t$. If $D \leq k$ almost surely, then there can be at most k t-bad points in the ball $B(0, t/2)$. Hence

$$k \geq \mathbb{E}[\text{number of } t\text{-bad points in } B(0, t/2)\}] = \text{vol}(B(0, t/2))\, \mathbb{P}^*(M > t),$$

giving the result. $\qquad\square$

Proof of Proposition 1.6. Assume that the number of edges that cross the origin is finite with positive probability. On the event that the origin is crossed by finitely many edges, the same is true for any other $x \in \mathbb{R}$, since the difference between the number of edges crossing x and the number of

edges crossing the origin is bounded above by the total degree of the vertices between the origin and x. Now consider the intervals between the points $x \in [\mathcal{P}]$ with odd degrees. When passing a point with odd degree, the number of crossing edges changes parity, that is, if it is even (odd) immediately to the left of the point, it is odd (even) to the right. When passing a point with even degree on the other hand, the parity of the number of crossing edges remains unchanged. This means that we can assign the value 0 (even number of edges crossing) or 1 (odd number of edges crossing) to the intervals separating the odd degree vertices in a deterministic way (indeed, the stable multi-matching is a factor). Furthermore, the odd degree vertices constitute a Poisson process. Now, [7, Lemma 11] asserts that it is impossible to assign alternating values 0 and 1 to the intervals separating the points of a Poisson process as a factor of the Poisson process. Here we need the stronger statement that this cannot be done even using the randomness in the degrees of the vertices and in the position of the even degree vertices. This however follows from a straightforward modification of the proof of [7, Lemma 11]. □

4 Percolation for the stable multi-matching with $D \equiv 2$

If the stable multi-matching almost surely has an infinite component, then there is a strictly positive probability p that a given vertex belongs to this component. Simulations of the stable multi-matching with $D \equiv 2$ on large finite cycles indicate a largest component comprising about 0.3 of the vertices (see the top row of Table 1 in Section 6 below). This suggests the existence of an infinite component with $p \approx 0.3$. In this section we show that percolation indeed follows from the assumption that a certain finite event has sufficiently large probability. Furthermore, we give overwhelming statistical evidence for this assumption.

The key concept for the proof is the **core (stable) multi-matching**, which we define next (in the more general setting of arbitrary dimension and numbers of stubs). Let $S \subset \mathbb{R}^d$ be a bounded set, let $P \subset S$ be a finite set of points, and let $(D_x)_{x \in P}$ be positive integers representing numbers of stubs. Let $\widetilde{P} = P \cup \{S^C\}$, where $S^C := \mathbb{R}^d \setminus S$. (We will treat S^C like an additional point; it will not form part of the matching, but will affect the

notion of closest points.) Assume that all distances between pairs of elements of $\widetilde{P}$ are distinct. Assign D_x stubs to each point $x \in P$, and one stub to S^C. Repeat the following operations. From each point $x \in P$ that currently has an unused stub, assign an arrow pointing to the closest other element of $\widetilde{P}$ among those that have at least one unused stub and do not already have an edge to x. Then, for every pair $x, y \in P$ whose arrows point to each other, connect them with an edge and remove one stub from each. Erase all arrows and repeat. After some finite number of such iterations, no more edges are added. The core multi-matching of (P, D) in S defined to be the resulting graph. Note that the degree of a vertex $x \in P$ is at most D_x, but may be strictly less.

Lemma 4.1. *Let P be any discrete set of points in $\mathbb{R}^d$, let $(D_x)_{x \in P}$ be positive integers, and let S be a bounded set. Every edge in the core multi-matching of $(P \cap S, D)$ in S is present in every stable multi-matching of (P, D) on $\mathbb{R}^d$.*

Proof. It is straightforward to check by induction on the steps of the above algorithm that every edge added is present in any stable multi-matching. The key point is that if $x \in S$ is closer to some other point in $y \in S$ than to S^C, then x is also closer to y than to any point in $P \setminus S$. $\qquad\square$

We now specialize to the main case of interest. Let $d = 1$, and let $S = I = [a, b]$, a bounded interval. Let $\mathcal{P}$ be a Poisson process of intensity 1 on $\mathbb{R}$, and consider the case $\mu(\{2\}) = 1$ of deterministically two stubs per vertex. By the core multi-matching on the interval I we mean the core multi-matching of $([\mathcal{P}] \cap I, D)$ on I, where $D \equiv 2$. We call an interval $I = [a, b]$ **good** if the core matching on I has a connected component with a point in the first quarter $[a, \frac{3}{4}a + \frac{1}{4}b]$ and a point in the last quarter $[\frac{1}{4}a + \frac{3}{4}b, b]$.

Theorem 4.2. *Let $\mathcal{P}$ be a Poisson process of intensity 1 on $\mathbb{R}$ and let $\mu(\{2\}) = 1$. If for some L we have $\mathbb{P}([0, L]$ is good$) > 0.968$, then the 2-stub stable multi-matching has an infinite component.*

Monte-Carlo simulations provide overwhelming evidence that the condition in Theorem 4.2 indeed holds for some large L, subject to the trustworthiness of the pseudo-random number generator and the software used. Indeed, in 1000 independent simulation runs of the process with $L = 13000$, the interval $[0, L]$ was good in 991 cases, implying that the hypothesis that the probability is 0.968 or less can be rejected at the 10^{-6} level. See the appendix for details.

By a **monotone path** in a multi-matching we mean a sequence of vertices $x_1 < x_2 < \cdots < x_k$ with the edges $(x_1, x_2), (x_2, x_3), \ldots, (x_{k-1}, x_k)$ all present. As observed in the proof of Theorem 1.3 (ii), no two edges cross in the 2-stub stable multi-matching, and hence the same holds in a core multi-matching. If I is good, it follows that the core multi-matching on I contains a monotone path from the first quarter to the last quarter. We call such a path a **spanning path** of the good interval.

Lemma 4.3. *Let $a < b < c < d$ be points of $\mathcal{P}$, and suppose that the intervals $[a, b]$ and $[c, d]$ are both longer that $[b, c]$. If the 2-stub stable multi-matching has a monotone path α from a to b and a monotone path δ from c to d, then it has a monotone path from a to d which contains α and δ.*

Proof. Suppose on the contrary that there is no monotone path from a to d containing α and δ. First extend the path α to the right as far as possible within $[b, c]$; that is, let $b' \in [b, c]$ be as large as possible such that there is a monotone path containing α from a to b'. Similarly extend γ as far left as possible to $c' \in [b, c]$. By our assumption, and since there are no crossing edges, we have $b' < c'$. Note also that $[a, b']$ and $[c', d]$ are longer than $[b', c']$. Now b' is adjacent to its neighbour in the monotone path from a, and to exactly one other vertex x. By our assumptions, $x \notin [b', c']$, and therefore $x \notin (a, d)$, otherwise we would have crossing edges. A similar argument shows that c' has a neighbour outside (a, d). But now (b', c') form an unstable pair. $\square$

Corollary 4.4. *If at least 8 of the 9 intervals $[0, x], [x, 2x], \ldots, [8x, 9x]$ are good, then so is $[0, 9x]$. Furthermore, under the same assumption, given any spanning paths, one of each of the good short subintervals, there is a spanning path of the long interval containing all of them.*

Proof. Let the configuration outside $I := [0, 9x]$ be arbitrary and consider the stable multi-matching. Write $I_k = [(k-1)x, kx]$. For any sequence of consecutive good intervals $I_a, I_{a+1}, \ldots, I_b$, by Lemma 4.3 we obtain a monotone path in their union reaching to within distance $x/4$ of each end. If I_3, I_4, I_5, I_6, I_7 are all good, then the resulting path reaches to within $2x + x/4 = 9x/4$ of each end of I, as required. On the other hand, if one of $I_3, I_4, \ldots, I_7$ is bad (but the other 8 subintervals are good), then we obtain two paths of length greater than $2x - 2(x/4) = 3x/2$ on either side of the bad subinterval, with a gap of length less than $x + 2(x/4) = 3x/2$ in between, so another application of Lemma 4.3 provides the required spanning path. $\square$

Proof of Theorem 4.2. Let

$$I^k := \left[-\frac{9^k L}{2}, \frac{9^k L}{2} \right],$$

and let p_k be the probability that I^k is good. Thus $p_0 > 0.968$, and by Corollary 4.4, $p_{k+1} \geq f(p_k)$ where

$$f(p) := p^9 + 9p^8(1 - p).$$

It follows by an elementary computation that $p_k \to 1$ as $k \to \infty$, and indeed $\sum_k (1 - p_k) < \infty$. Hence by the Borel-Cantelli lemma, a.s. I^k is good for all sufficiently large k. Moreover, since $\sum_k (1 - f(p_k)) < \infty$, for all sufficiently large k, the interval I^k can be divided into 9 equal intervals of which at least 8 are good. By Corollary 4.4 it follows that for some (random) K we may find monotone paths $\pi_K, \pi_{K+1}, \pi_{K+2}, \ldots$, each contained in the next, with π_k a spanning path of I^k for each k. Then $\bigcup_{k \geq K} \pi_k$ is an infinite connected graph in the stable multi-matching. $\square$

5 Some counterexamples

Theorem 1.2 asserts that the random direction stable multi-matching has no infinite component when all vertices have degree 2, and that it has long edges in the sense that $\mathbb{E}^*[X^{1/2}] = \infty$. Furthermore, it follows from Theorem 1.3 that existence of an infinite component in the stable-multi matching with all degrees equal to 2 is equivalent to $N < \infty$. This might lead one to suspect that there is a simple relation between the component structure and the edge length for $\mu(\{2\}) = 1$ that holds for any matching scheme. Below, we give two examples of factor matching schemes that demonstrate that this is not the case.

Example 1. Our first example is a matching scheme where all components are infinite and where also the number of edges crossing the origin is a.s. infinite. Note that, if the origin is crossed by infinitely many edges, then also $N = \infty$ and thus, by Lemma 1.4, $\mathbb{E}^*[M] = \infty$. Existence of an infinite component hence does not imply short edges in any of these respects.

To describe the matching scheme, let each point in $[\mathcal{P}]$ be equipped with two stubs. Recall that the stable multi-matching in the special case where

$\mu(\{1\}) = 1$ is known as the stable matching. First use one stub per point to form edges according to the stable matching of the points. Then orient the remaining stub at each point in the opposite direction (left or right) from that of the first stub, and connect these directed stubs according to the procedure used for the random direction stable matching. This gives a graph where each point has one edge connected to the right and one edge connected to the left – that is, all points are birds in the in the terminology used in the proof of Theorem 1.2 – which implies that all components in the graph are infinite. That the number of edges crossing the origin is infinite almost surely follows from Proposition 1.6 applied to the configuration after the first stub per point is connected. □

Example 2. The next example is a matching scheme that gives almost surely only finite components and where the expected edge length is finite. Finite expected edge length hence does not imply existence of an infinite component.

The matching scheme proceeds by dividing the vertices into groups of size at least 3 as follows. Call a point of $\mathcal{P}$ a **seed** if it has some other point within distance 1. Call a seed x **good** if the number of non-seed points between x and the next seed to its right is at least 2. Now whenever $x < y$ are two consecutive good seeds, let all the points in $[x, y)$ constitute one group.

Define the matching as follows. For a group $x_1 < \cdots < x_k$, connect the two stubs per vertex to form the edges (x_i, x_{i+1}) for $i = 1, \ldots, k-1$ and the edge (x_1, x_k). This clearly gives a configuration with almost surely finite components and finite expected edge length. □

Next we give simple examples of translation-invariant point processes on $\mathbb{R}$ for which the stable multi-matching in the case $\mu(\{2\}) = 1$ provably does, and does not, have an infinite component.

Example 3. Let $(X_i)_{i \in \mathbb{Z}}$ be i.i.d. and uniformly distributed on $[0, 1/3]$, and let U be independent and uniform on $[0, 1]$. Consider the point process with support $\{i + X_i + U : i \in \mathbb{Z}\}$. It is easy to see that each point connects to its left-neighbour and its right-neighbour, so there is an infinite component.

Example 4. Let $(X_{i,j})_{i \in \mathbb{Z}, j=1,2,3}$ be i.i.d. and uniformly distributed on $[0, 1/3]$, and let U be independent and uniform on $[0, 1]$. Consider the point process with support $\{i + X_{i,j} + U : i \in \mathbb{Z}, j = 1, 2, 3\}$. Then each component has size exactly 3.

		Number of points				
		2^{10}	2^{12}	2^{14}	2^{16}	2^{18}
Expected degree	2	$.244 \pm .099$	$.291 \pm .044$	$.297 \pm .014$	$.287 \pm .009$	$.292 \pm .005$
	3	$.278 \pm .099$	$.154 \pm .029$	$.049 \pm .006$	$.017 \pm .003$	$.006 \pm .001$
	4	$.802 \pm .158$	$.728 \pm .232$	$.653 \pm .201$	$.366 \pm .207$	$.399 \pm .143$
	5	$.974 \pm .018$	$.933 \pm .114$	$.815 \pm .183$	$.672 \pm .258$	$.321 \pm .102$
	6	$.989 \pm .009$	$.990 \pm .004$	$.975 \pm .047$	$.933 \pm .161$	$.755 \pm .207$
	7	$.994 \pm .008$	$.997 \pm .003$	$.989 \pm .020$	$.996 \pm .000$	$.961 \pm .112$
	2.1	$.071 \pm .023$	$.024 \pm .005$	$.009 \pm .002$	$.003 \pm .000$	$.001 \pm .000$
	2.5	$.132 \pm .062$	$.043 \pm .018$	$.018 \pm .006$	$.006 \pm .001$	$.001 \pm .000$
	3.5	$.472 \pm .172$	$.244 \pm .071$	$.146 \pm .061$	$.050 \pm .014$	$.016 \pm .004$
	4.5	$.992 \pm .011$	$.888 \pm .110$	$.529 \pm .138$	$.298 \pm .132$	$.129 \pm .059$

Table 1: Simulation results for the stable multi-matching of uniformly random points on the cycle. The proportion of points in the largest connected component is indicated as "sample mean $\pm$ sample standard deviation" for a sample of size 10. The degree D is either a constant integer, or takes two consecutive integer values with probabilities determined by the indicated expected value.

6 Open problems

The random direction stable multi-matching

For degree distributions other than $\mu(\{2\}) = 1$, it remains an open problem to determine if the random direction stable multi-matching generates an infinite component.

The stable multi-matching

Firstly, it would of course be desirable to turn the "statistical proof" of percolation for $D \equiv 2$ in $d = 1$ into a fully rigorous proof. Furthermore, it remains an open problem to determine whether there exists an infinite component the stable multi-matching for other degree distributions (an exception being the case $D \in \{1, 2\}$ with $\mathbb{P}(D = 1) > 0$; see [3]). Another interesting case arises when most vertices have 2 stubs, but a small fraction have 3; this case can be expected to be very different from the 2-stub case since there are local configurations which can end a long path. Indeed, simulations appear to

indicate that the proportion of vertices in the largest component converges to 0 as the system size increases, thus suggesting no infinite component; see Table 1 (lines 9-10). The observation that local modifications can end a long path extends to any degree distribution with support on at least one odd integer. Is it the case that the stable multi-matching has an infinite component in $d = 1$ if and only if the degree distribution has support only on even integers? The results in Table 1 appear consistent with this hypothesis.

Iterated stable matching

Yet another multi-matching scheme is obtained by repeatedly applying the stable matching of the points with the restriction that multiple edges are not allowed. More specifically, first take the stable matching of $[\mathcal{P}]$, connect the points accordingly and remove one stub per point. Then consider the stable matching of the points that have at least one stub left on them and with the modification that two points that already have an edge between them cannot be matched. This matching is obtained by repeatedly matching mutually nearest neighbors in the set of points with at least one stub left on them, avoiding matchings of points that are already connected. As remarked in [3, Remark 2.2], the proof of [3, Proposition 2.2] is easily modified to show that this yields a perfect matching. Connect the points according to this matching and remove one stub from each point that is connected. Repeat indefinitely.

Does this matching scheme generate an infinite component? How does the answer depend on the degree distribution? Note that it follows from Proposition 1.6 that the number of edges that cross the origin is a.s. infinite already after the first stub of the vertices has been connected. For degree distributions with degrees larger than 1 however the matching contains crossing edges. This means for instance that the proof of Theorem 1.3(b) cannot be applied to draw the same conclusion (that $N < \infty$ if there is an infinite component) for the iterated stable matching.

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Appendix − simulation code

The Python 2.6 code below was used to verify that the interval $[0, 13000]$ was a good interval in 991 out of 1000 cases. Since

$$\mathbb{P}\big[\text{Binomial}(1000, 0.968) \geq 991\big] < 10^{-6},$$

this gives grounds to reject the hypothesis that $\mathbb{P}([0, 13000]$ is good$) \leq 0.968$ at the 10^{-6} level.

The code uses Python's built-in implementation of the Mersenne Twister, one of the most extensively tested pseudo-random number generators. The experiment was also repeated using two other pseudo-random number generators, and using an alternative method of generating a Poisson point process, with consistent results.

The following observations simplify and speed up the construction of the core multi-matching in the 2-stub 1-dimensional case. In place of the complement of an interval $[a, b]$, it suffices to consider distances to the endpoints

$\{a, b\}$. At any stage of the core multi-matching algorithm, the arrow from a point x with an unused stub must point to one of x's two closest neighbours on the left or its two closest neighbours on the right among the set of points of $\widetilde{P}$ with unused stubs; this is because at most one other point can have an edge to x already. Also, if the arrows of x, y point to each other, then there is no other point z with unused stubs between them; indeed such a z cannot already have edges to both x and y, so one of x and y would instead point to z.

```
from random import *
from math import *

def poi(a):        # Poisson random variable with mean a
    t=-1
    while a>0:
        a+=log(random())
        t+=1
    return t

def setup(a):
    global n,x,stubs,e
    n=poi(a)
    x=[random() for i in xrange(n)]
    x.sort()        # x = sorted list of n=poi(a) random points in [0,1]
    stubs=[2]*n     # 2 stubs per point
    e=set([])       # e = set of edges

def xx(i):         # position of point i, or endpoint of [0,1] for i outside range
    if i<0:
        return 0.
    elif i>=n:
        return 1.
    else:
        return x[i]

def corematch():
    cont=True
    while cont:
        cont=False
        active=[-1,-1]+[i for i in xrange(n) if stubs[i]]+[n,n]
                        # points with unused stubs, plus 2 dummy points at each end
        arrow=[None]*len(active)
        for j in xrange(2,len(active)-2):
            l=[(abs(xx(active[j])-xx(active[k])),k)      # find distances
                for k in j-2,j-1,j+1,j+2                  # to 2 neighbours on each side
                if tuple(sorted([active[j],active[k]])) not in e]  # if no edge already
            if l:
                arrow[j]=min(l)[1]                       # arrow to closest
        for j in xrange(2,len(active)-3):
            if arrow[j]==j+1 and arrow[j+1]==j:          # found pair with mutual arrows
                                                         # (must be neighbours)
                e.add((active[j],active[j+1]))           # add edge and remove stubs
                stubs[active[j]]-=1
```

```
                stubs[active[j+1]]-=1
                cont=True                              # keep going if some edge added

def components():                                      # find components of graph
    nbrs=dict((i,[]) for i in xrange(n))
    for (i,j) in e:
        nbrs[i].append(j)
        nbrs[j].append(i)
    done=[False]*n
    ans=[]
    for i in xrange(n):
        if not done[i]:
            cur=[i]
            done[i]=True
            for j in cur:
                for k in nbrs[j]:
                    if not done[k]:
                        cur.append(k)
                        done[k]=True
            ans.append(sorted(cur))
    return ans

def good():                                            # is the interval good?
    return any(x[c[0]]<.25 and x[c[-1]]>.75 for c in components())

seed(12345)
a=13000
k=1000
g=0
for i in xrange(k):
    setup(a)
    corematch()
    if good():
        g+=1
    print g,'/',(i+1),'..',
print
print 'Interval of length',a,'was good',g,'times out of',k
```