ROTOR WALKS ON GENERAL TREES

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Abstract. The rotor walk on a graph is a deterministic analogue of random walk. Each vertex is equipped with a rotor, which routes the walker to the neighbouring vertices in a fixed cyclic order on successive visits. We consider rotor walk on an infinite rooted tree, restarted from the root after each escape to infinity. We prove that the limiting proportion of escapes to infinity equals the escape probability for random walk, provided only finitely many rotors send the walker initially towards the root. For i.i.d. random initial rotor directions on a regular tree, the limiting proportion of escapes is either zero or the random walk escape probability, and undergoes a discontinuous phase transition between the two as the distribution is varied. In the critical case there are no escapes, but the walker's maximum distance from the root grows doubly exponentially with the number of visits to the root. We also prove that there exist trees of bounded degree for which the proportion of escapes eventually exceeds the escape probability by arbitrarily large $o(1)$ functions. No larger discrepancy is possible, while for regular trees the discrepancy is at most logarithmic.

1. Introduction

The **rotor walk** is a derandomized variant of the random walk on a graph G, defined as follows (see also [12, 13], for example). To each vertex of G we assign a fixed cyclic order of its neighbours, and at each vertex there is a **rotor**, which points to some neighbour. A **particle** is located at some vertex. The particle location and rotor positions evolve together in discrete time as follows. At each time step, the rotor at the particle's current location is first incremented to point to the next neighbour in the cyclic order, and then the particle moves to this new neighbour. The rotor walk is obtained by repeatedly applying this rule.

In many settings, there is remarkable agreement between the behaviour of rotor walk and expected behaviour of *random* walk; see for

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example [3, 5, 6, 13, 15, 21, 19, 20]. However, the two processes can also exhibit striking differences. Landau and Levine [17] demonstrated this in the context of recurrence and transience properties for regular trees (see also [2]). In this article we further investigate these issues for general trees.

Suppose that G is an infinite graph with all degrees finite. Let the particle start at a fixed vertex o, called the **root**, and consider the rotor walk stopped at the first return to o. Either the particle returns to o after a finite number of steps, or it escapes to infinity without returning to o, visiting each vertex only finitely many times. In either case, the positions of the rotors after the walk is complete are well defined. We then start a new particle at o (without changing the rotor positions), and repeat the above procedure, and so on indefinitely. Let E_n be the number of particles that escape to infinity (as opposed to returning to o) after n rotor walks are run from o in this way (so $0 \leq E_n \leq n$).

Let $\mathcal{E} = \mathcal{E}(G)$ denote the probability that a simple symmetric random walk on G started at o never returns to o. The following key result of Schramm, a proof of which is presented in [13, Theorem 10], states that the rotor walk is in a certain sense no more transient than the random walk.

Theorem 1 (Density bound; Oded Schramm)**.** *For any graph with all degrees finite, any starting vertex o, any cyclic orders of neighbours and any initial rotor positions,*

$$\limsup_{n\to\infty} \frac{E_n}{n} \leq \mathcal{E}.$$

The above result suggests the following questions. When does E_n/n have a limit, and when is it equal to $\mathcal{E}$? When is it strictly smaller? In the former case, how large can the difference $E_n - \mathcal{E}n$ be, as a function of n? The answers in general depend both on the graph G and the initial positions of the rotors. We will provide answers when the graph is a tree.

Let $G = T$ be an infinite tree with vertex set V and all degrees finite. For notational convenience, we take T to denote the abstract tree *together with the cyclic orders* of neighbours. We will also always assume that the root o has degree 1 (if this condition is dropped, it is straightforward to reformulate all our results appropriately). For a vertex $v \neq o$, let $v^{(0)}$ be its **parent** (i.e. its neighbour on the unique path to the root), and let $v^{(1)}, \dots, v^{(b)}$ be its $b = b(v)$ **children** (i.e. its remaining neighbours), labeled so that the rotor at v points to the neighbours in the cyclic order $v^{(0)}, v^{(1)}, \dots, v^{(b)}$. (However, we will allow it to start at an arbitrary point in this order.) Write $V_o := V \setminus \{o\}$ and

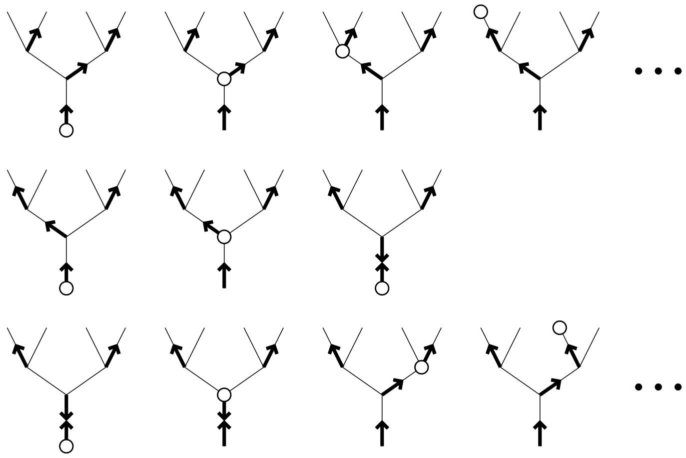

FIGURE 1. The rotor walk on an infinite binary tree (shown up to level 3). The rotors (thick arrows) turn anticlockwise with positions 0 (down), 1 (right), 2 (left), and all non-root rotors have initial position 1. The first particle (white disc) escapes to infinity, the second returns to the root, and the third escapes, so $E_3 = 2$.

$\mathbb{N} = \{0, 1, 2 \ldots\}$. A **rotor configuration** is a map $r : V_o \to \mathbb{N}$, with $0 \le r(v) \le b(v)$ for all v, indicating that the rotor at vertex v points to $v^{(r(v))}$ (the rotor at o always points to its only child). Our principal object of study is the escape number $E_n = E_n(T, r)$, defined to be the number of particles that escape to infinity after n successive rotor walks are run from o (as discussed above), with initial rotor configuration r. See Figure 1.

Let $\mathbf{0}$ denote the rotor configuration in which each rotor points towards the root, and let $\mathbf{b}$ denote the rotor configuration in which each rotor will *next* point towards the root, i.e. let $\mathbf{0}(v) = 0$ and $\mathbf{b}(v) = b(v)$ for all $v \in V_o$.

Theorem 2 (Monotonicity, extremal configurations). *Let T be a tree.*

(i) *If r, s are rotor configurations satisfying $r(v) \le s(v)$ for all $v \in V_o$, then $E_n(T, r) \ge E_n(T, s)$ for all $n \ge 0$.*
(ii) *We have $E_n(T, \mathbf{0}) \ge \mathcal{E} n$ for all $n \ge 0$.*
(iii) *We have $E_n(T, \mathbf{b}) = 0$ for all $n \ge 0$.*

It follows immediately from Theorem 1 and Theorem 2(ii) that $\lim_{n\to\infty} E_n(T, \mathbf{0})/n = \mathcal{E}$. We establish the following much stronger result.

Theorem 3 (Limiting density)**.** *For any tree T, if r is any rotor configuration having only finitely many vertices v with $r(v) = b(v)$, then*

$$\lim_{n\to\infty} \frac{E_n(T,r)}{n} = \mathcal{E}.$$

In the case of the binary tree $\mathbb{T}_2$, Landau and Levine [17, Theorem 1.2] characterized all possible sequences $(E_n(\mathbb{T}_2, r))_{n\geq 0}$ that arise as the configuration r varies. Their result implies in particular that for any α, β satisfying $0 \leq \alpha \leq \beta \leq \mathcal{E}(= \frac{1}{2})$ there exists a configuration r such that $\liminf_{n\to\infty} E_n(\mathbb{T}_2, r)/n = \alpha$ and $\limsup_{n\to\infty} E_n(\mathbb{T}_2, r)/n = \beta$.

Theorem 1 leaves open the possibility that the number of escapes E_n exceeds the expected number $\mathcal{E}n$ for random walk by an arbitrarily large $o(n)$ function (but no more than this), and our next result shows that this is indeed possible, even for a tree of bounded degree.

Theorem 4 (Large discrepancy)**.** *For any function f that satisfies $f(n) = o(n)$ as $n \to \infty$, there exists a tree T of maximum degree 3 such that*

$$E_n(T, \mathbf{0}) \geq \mathcal{E}n + f(n) \qquad \text{for all } n \text{ sufficiently large.}$$

In fact we will prove Theorem 4 by exhibiting a recurrent tree with the required property (so $\mathcal{E} = 0$). It is easy to obtain an example with any given $\mathcal{E} \in (0, 1)$ by adjoining a suitable transient tree.

We next specialize to regular trees. For $b \geq 2$, let $\mathbb{T}_b$ be the **b-ary** tree, in which every vertex $v \in V_o$ has exactly b children. In contrast with Theorem 4, in this case the number of escapes differs from its expected value by at most a logarithmic term.

Theorem 5 (Regular trees)**.** *For the b-ary tree $\mathbb{T}_b$, let $\mathbf{k}$ be the constant configuration given by $\mathbf{k}(v) = k$ for all $v \in V_o$. For $b \geq 2$ and $0 \leq k < b$, the normalized discrepancy*

$$\Delta_n := \frac{E_n(\mathbb{T}_b, \mathbf{k}) - \mathcal{E}n}{\log_b n}$$

satisfies

$$\liminf_{n\to\infty} \Delta_n = \begin{cases} 0, & k = 0; \\ -\frac{k-1}{b}, & k > 0; \end{cases} \qquad \limsup_{n\to\infty} \Delta_n = \frac{b-k-1}{b}.$$

Recall that the remaining case $k = b$ is covered by Theorem 2(iii). The presence of the exceptional case $k = 0$ for the liminf is perhaps surprising. In particular, the difference $\limsup \Delta_n - \liminf \Delta_n$ equals $(b-2)/b$ for $k > 0$ but $(b-1)/b$ for $k = 0$. In the only case where this difference is zero, namely $(b, k) = (2, 1)$, we in fact have $E_n = \lceil n/2 \rceil$, as noted in [17]. The proof of Theorem 5 involves an explicit expression for $E_n(\mathbb{T}_b, \mathbf{k})$ for the general case, so further refinements of the bounds are available.

Now consider a *random* initial configuration R on the b-ary tree, in which $(R(v))_{v\in V_o}$ are independent identically distributed random variables having some distribution on $\{0, 1, \dots, b\}$. Let $\mathbb{P}$ denote probability and $\mathbb{E}$ expectation. Assume that the distribution is not deterministic (otherwise Theorems 2(iii) and 3 apply). Then the behaviour of the rotor walk depends dramatically on the distribution.

Theorem 6 (Discontinuous phase transition)**.** *For a non-deterministic i.i.d. rotor configuration R on the b-ary tree $\mathbb{T}_b$, writing $v(\neq o)$ for an arbitrary vertex, we have almost surely*

$$\text{(i) } \lim_{n\to\infty} \frac{E_n(\mathbb{T}_b, R)}{n} = \mathcal{E} \qquad \textit{if } \mathbb{E}R(v) < b-1;$$

$$\text{(ii) } E_n(\mathbb{T}_b, R) = 0,\ \forall n \geq 0 \qquad \textit{if } \mathbb{E}R(v) \geq b-1.$$

The dichotomy in Theorem 6 reflects the survival or otherwise of a certain branching process. In particular we note that the archetypal example of i.i.d. uniformly random rotors on the binary tree $\mathbb{T}_2$ (i.e. $\mathbb{P}(R(v) = i) = 1/3$ for $i = 0, 1, 2$) is critical ($\mathbb{E}R(v) = b - 1$), so there are no escapes. This answers a question of Jim Propp (personal communication). Note the contrast with the *deterministic* case $R(v) \equiv b - 1$, in which Theorem 3 shows that the proportion of escapes is $\mathcal{E}$.

Finally, we show that in the critical case, particles typically reach very great depths in the tree before returning to the root. This has implications for simulation: although all particles return to the root, this will not be evident in any finite simulation of practical size.

Theorem 7 (Maximum depth)**.** *Let R be a non-deterministic i.i.d. configuration on the b-ary tree $\mathbb{T}_b$, for $b \geq 2$. Let D_n be the maximum distance from the root reached by the first n particles. Then for some constants $c, C \in (0, \infty)$ depending on the distribution of R,*

$$\text{(i) } \mathbb{P}\big[e^{e^{cn}} < D_n < e^{e^{Cn}}\big] \to 1 \textit{ as } n \to \infty \quad \textit{if } \mathbb{E}R(v) = b-1;$$

$$\text{(ii) } \mathbb{P}\big[n \leq D_n < Cn\big] \to 1 \textit{ as } n \to \infty \quad \textit{if } \mathbb{E}R(v) > b-1.$$

Further Remarks. The rotor walk was first introduced in [22], under the name "Eulerian walkers model". For a general account of the model and its history, see [12]. In [13] it was shown that on finite graphs, and some infinite graphs, many quantities associated with random walk (hitting probabilities, hitting times, stationary measures, etc.) are very well approximated by rotor walk counterparts. In [4, 5] a different variant is studied, in which several rotor walks are run simultaneously for a fixed number of steps; again it is shown that this model closely approximates the expected behaviour of random walk. The rotor aggregation model (introduced by Propp) is a rotor version of internal diffusion-limited aggregation, in which successive rotor walks are started at a fixed vertex and run until they first reach a previously unoccupied vertex. On the integer lattice, the resulting occupied set rapidly approaches a spherical ball, as proved in [19, 20].

Rotor walks on trees have been studied in detail by Landau and Levine [17] (as well as earlier in [2, 8]). In particular, on the binary tree $\mathbb{T}_2$ they identified the escape sequences for the constant configurations $\mathbf{0}$ (no escapes) and $\mathbf{1}$ (alternating escapes and returns), and furthermore provided an exact characterization for the set of possible escape sequences $(E_n(\mathbb{T}_2, r))_{n\geq 0}$. This characterization is more conveniently stated in terms of the binary escape sequence $e = (e_1, e_2, \ldots)$ where e_n is the indicator that particle n escapes (so $E_n = \sum_{i=1}^{n} e_i$). The result of [17] is that e is an escape sequence for some rotor configuration on $\mathbb{T}_2$ if and only if for each $k \geq 2$, each interval of e of length $2^k - 1$ contains at most 2^{k-1} ones. In particular it follows that if e is a possible escape sequence, then so is any e' satisfying $e'_n \leq e_n$ for all n (this implies that E_n/n may have arbitrary liminf and limsup in $[0, 1/2]$, as remarked earlier). We do not know whether this holds for general trees.

The rotor aggregation model on regular trees was also studied in [17], where it was shown that the occupied set is exactly a ball for acyclic initial rotor configurations. The proof makes use of the sandpile group (see also [12, 18]). We remark that the rotor aggregation model on a tree T can be encoded as an ordinary rotor walk process on a different tree T'. We construct T' by attaching a singly infinite path to each vertex of T, with all the rotors on the new vertices pointing towards T, and each rotor at an original vertex of T set to point next along the added path, and then in the direction it would have pointed next in T. When a particle reaches a vertex of T for the first time, it immediately escapes along the path (signifying an aggregation). On subsequent visits to the vertex, the particle makes only finite excursions along the path, and then continues as the rotor walk on T would have done.

We briefly mention two somewhat degenerate classes of trees which we will not focus on. First, suppose some vertices have no children (this is not forbidden by our assumptions). Then we may remove every vertex that has only finitely many descendants without affecting the behaviour of the rotor walk or the random walk, so there is no loss generality in discarding this case. Second, suppose a vertex w has infinitely many children (which is forbidden by our assumptions). Under the mild condition that the rotor at w will point to infinitely many children whose subtrees have the property that the first particle sent there escapes, then w acts as a *sink*, which is to say that every particle sent there escapes (and no other information about w's descendants is relevant). Finite trees with sinks are considered in Section 3, and many of our results could be extended without difficulty to infinite trees with sinks, but we have chosen not to pursue this.

Some applications of rotor walks may be found in [7, 8, 10], and further recent progress is reported in [11, 14, 21].

A key tool for our proofs will be a recursive formula for the escape numbers $E_n(T, r)$, called the *explosion formula*, which generalizes ideas in [17] and is introduced in Section 5. This will be used to prove monotonicity (Theorem 2(i)), the bounds for regular trees (Theorem 5), and a number of other results. The large-discrepancy tree in Theorem 4 will constructed from *brushes* (higher-dimensional analogues of combs), which we will analyze in Section 7. Theorem 3 will use a martingale argument for random walks on trees (Section 10). As remarked earlier, the discontinuous phase transition in Theorem 6 is related to a branching process. A simple argument will show that there are no escapes in the subcritical and critical cases (ii) (see Section 2); for the supercritical case (i), the proof makes use of the Abelian property (Section 9). The doubly-exponential depth in the critical case, Theorem 7(i), reflects the fact that the set of visited vertices is (a minor variation of) a branching process whose offspring distribution is itself the total population size of a critical branching process; see Section 11.

We will prove several other results in addition to those mentioned above, including: an expression for the total number of escapes (Proposition 8); Lipschitz, monotonicity, and invariance conditions for the escape numbers in terms of rotors, trees, and rotor orders (Propositions 12, 13, and 14); a self-majorization property (Proposition 16); and a surprising regularity result for the behaviour of rotor walks on finite regular trees (Proposition 22) with an arbitrary rotor configuration.

2. Live paths

In this section we prove Theorem 2(iii) and Theorem 6(ii). We introduce the concept of live paths, which will also be important for the later proof of Theorem 6(i).

Proof of Theorem 2(iii). With initial configuration **b**, suppose for a contradiction that some particle escapes to infinity. Let particle n be the first such particle. Clearly there exists a vertex v such that particle n visits both v and one of its children, and v was not visited by any previous particle. Since on the first visit to v the particle is sent back towards the root, particle n must move from the parent $v^{(0)}$ to v at least twice.

Let u be the first vertex to receive particle n twice from its parent. The parent cannot be o, since by assumption particle n never returns to o. Hence the rotor at the parent $u^{(0)}$ must have made a full rotation between the two traversals of particle n from $u^{(0)}$ to u. In particular, $u^{(0)}$ sent the particle to its parent in between, and thus $u^{(0)}$ received the particle twice from its parent before u did, a contradiction. □

Given a tree T and a rotor configuration r, a **live path** is an infinite sequence of vertices $o \neq x_1, x_2, \ldots$, each the parent of the next, such that for each i, the k such that $x_{i+1} = (x_i)^{(k)}$ satisfies $r(x_i) < k$ – in other words, x_i will send the particle forward to x_{i+1} before sending it back towards the root. An **end** of T is an infinite sequence of vertices $o = x_0, x_1, \ldots$, each the parent of the next. Call an end **live** if the subsequence $(x_i)_{i \geq j}$ starting at one of its vertices is a live path. Let $E_\infty(T, r) := \lim_{n\to\infty} E_n(T, r)$ be the total number escapes ever (which may be a non-negative integer or ∞).

Proposition 8 (Live ends)**.** *The total number of escapes $E_\infty(T, r)$ equals the number of live ends in the initial rotor configuration r.*

Proof. Suppose that $o \neq x_1, x_2, \ldots$ is a live path, and that the particle is currently located at x_1. We claim that the particle will escape to infinity without ever visiting the parent of x_1. Suppose on the contrary that it visits the parent of x_1 at some (finite) time. By the definition of a live path, it must previously have visited x_2. Similarly, considering the rotor at x_2, we deduce that it must have visited x_3 before moving from x_2 to x_1. Repeating this argument shows that the particle must have visited the infinite set of vertices $x_2, x_3, \ldots$ before the parent of x_1, a contradiction which proves the claim.

Now suppose that the set of live ends is L, and a particle is started at o. If the particle returns to o then the set of live ends is still L at this point, because only finitely many vertices have been visited. Suppose on the other hand that the particle escapes without returning to o. The set of all vertices visited by this particle consists of exactly one end $\eta = (x_0, x_1, \ldots)$, together with finite trees rooted at vertices of η. We claim that the particle never backtracks on η, i.e. after its first visit to x_{i+1}, it never visits x_i. This holds for $i = 1$ by assumption. Suppose that it fails for some $i > 1$, i.e. the particle visits x_i, then x_{i+1}, then x_i again. Before another visit to x_{i+1} (which must eventually occur), the rotor at x_i must make a full rotation, so the particle must visit x_{i-1}, implying that the claim fails for $i-1$ also. Thus, the non-backtracking claim is proved. It follows that $(x_1, x_2 \ldots)$ was originally a live path, and in particular $\eta \in L$. Just after the particle has escaped, all rotors on η point along η away from the root, so η is no longer live; however, any other end has been visited at only finitely many of its vertices; consequently the set of live ends is now $L \setminus \{\eta\}$.

Finally observe that, in the infinite sequence of restarts at o implicit in the definition of E_∞, each vertex is visited infinitely often (this holds for the child of o, therefore for each of its children, etc.).

The result now follows from the above claims. So long as there are live ends, there will eventually be a visit to a live path and an escape, which results in the set of live ends being depleted by one; at all other times this set remains unchanged. □

As usual, if vertex u is on the unique self-avoiding path from v to o, then we say that u is an **ancestor** of v, and v is an **descendant** of u.

Proof of Theorem 6(ii). Suppose that R is an i.i.d. random configuration. Let $u \neq o$ be a vertex and let $v = u^{(k)}$ be one of its children. Call the child v **good** if $R(u) < k$. Thus, $x_1, x_2, \ldots$ is a live path if and only if $x_2, x_3, \ldots$ are all good. On the other hand, the number of good children of u is precisely $b - R(u)$. Hence, for any fixed $u \neq o$, the set of descendants w of u that can be reached from u via a path of good vertices (including w but not necessarily u) forms a Galton-Watson branching process with offspring distribution that of $b - R(u)$. If R is non-deterministic and $\mathbb{E}R(v) \geq b - 1$ then this process dies out, hence a.s. there are no live paths, and therefore no escapes by Proposition 8. □

3. Truncation

It will sometimes be useful to approximate infinite trees with finite ones. Let T be a *finite* tree with a root o of degree 1, and let $S \not\ni o$ be a non-empty subset of the leaves; we call elements of S **sinks**. Consider a rotor walk with initial configuration r, started at o and stopped at the first entry to $S \cup \{o\}$. Let $E_n(T, S, r)$ be the number of particles that stop at S when n such walks are run in succession, starting with configuration r.

The **level** of a vertex is its graph-distance from the root. Given an infinite tree T and a positive integer h, define the truncated tree T^h to be the subgraph induced by the set of vertices at levels at most h, and let S^h be the set of vertices at level exactly h. Also let r^h denote the rotor configuration r restricted to the vertices of T^h (excluding S^h and o).

Lemma 9 (Truncation)**.** *For any tree T, any rotor configuration r, and any n,*

$$E_n(T^h, S^h, r^h) \searrow E_n(T, r) \quad \text{as } h \to \infty,$$

(i.e. the left side is decreasing in h, and equals the right side for h sufficiently large).

Lemma 9 is proved in [13, Lemmas 18,19]; indeed it holds for arbitrary graphs, and in the more general setting of rotor walks associated with Markov chains. For the reader's convenience we summarize the argument here. The convergence follows from the stronger statement that the number of visits to any given vertex converges similarly, which is proved by induction on n. Monotonicity is a consequence of the Abelian property for rotor walks on a finite graph with a sink (see e.g. [12]).

4. Rotors pointing towards the root

In this section we will prove Theorem 2(ii). We will prove a version for finite trees and then appeal to Lemma 9.

Consider a finite tree T in which the root o has exactly one child, and let $S \not\ni o$ be a non-empty subset of the leaves of T. Construct a directed graph $G = G(T, S)$ as follows. Replace each edge of T not incident to S with two directed edges, one in each direction. Replace each edge incident to S with a single directed edge towards S. Add directed edges from each vertex in S to o. Note that if a particle performs some walk on T and is returned to o whenever it reaches S, then its trajectory is a (directed) path in G.

Consider any finite directed path π in the graph G, starting and ending at the same vertex. Let $v \neq o$ be any vertex of the original tree T, and let $u = v^{(0)}$ be its parent. Let n_v be the number of times the path traverses the directed edge from u to v, and m_v the number of traversals from v to u. Let p_v be the probability that the simple random walk on T started at v hits u before S. Define the discrepancy

$$\delta_v := p_v n_v - m_v.$$

Lemma 10. *Fix a finite tree and a path π as above, and let $v \notin S \cup \{o\}$ be a vertex, with children $v^{(1)}, \ldots, v^{(b)}$. Abbreviate δ_v to δ, and $\delta_{v^{(i)}}$ to δ_i, and similarly for n, m, and p. We have*

$$\delta = p \sum_{i=1}^{b} \big[\delta_i + (1 - p_i)(n_i - m)\big].$$

Proof. First consider the random walk started at v. At each visit to v, the walk either returns immediately to the parent with probability $1/(b+1)$, or escapes to S via $v^{(i)}$ with probability $(1-p_i)/(b+1)$, or otherwise returns to v without doing either. Therefore the probability p of returning to the parent before escaping to S is given by

$$p = \frac{1/(b+1)}{1/(b+1) + \sum_{i=1}^{b}(1-p_i)/(b+1)} = \frac{1}{1 + \sum_{i=1}^{b}(1-p_i)}. \tag{1}$$

Turning to the path π, since the numbers of arrivals and departures at v are equal we have $n + \sum_{i=1}^{b} m_i = m + \sum_{i=1}^{b} n_i$, or equivalently,

$$\sum_{i=1}^{b}(n_i - m_i) = n - m. \tag{2}$$

Now using the definition of δ_i, followed by (2), (1), and the definition of δ, we obtain

$$\begin{aligned}\sum_{i=1}^{b} \big[\delta_i + (1-p_i)(n_i - m)\big] &= \sum_{i=1}^{b} \big[(n_i - m_i) - m(1-p_i)\big] \\ &= n - m - m\sum_{i=1}^{b}(1-p_i) = n - m/p = \delta/p,\end{aligned}$$

as required. □

Proof of Theorem 2(ii). Apply Lemma 10 to a rotor walk on a finite tree T with initial configuration $\mathbf{0}$, started at o, returned to o after every visit to S, and run up until some visit to o. In this case, considering the rotor at v shows that $n_i \geq m$, therefore the lemma gives $\delta \geq \sum_{i=1}^{b} \delta_i$. For any $w \in S$ we have $p_w = m_w = 0$, therefore $\delta_w = 0$. Hence by applying the previous inequality iteratively we deduce that $\delta_\iota \geq 0$, where ι is the child of the root. In other words, for a finite tree, $E_n(T, S, \mathbf{0}) \geq \mathcal{E}(T, S)\, n$, where $\mathcal{E}(T, S) = 1 - p_\iota$ is the probability that a random walk started at o hits S before returning to o.

Now for an infinite tree T we apply Lemma 9. By the above we have for all n and h,

$$E_n(T^h, S^h, \mathbf{0}) \geq \mathcal{E}(T^h, S^h)\, n,$$

while
$$\mathcal{E}(T^h, S^h) \searrow \mathcal{E}(T) \quad \text{as } h \to \infty.$$
Hence the lemma gives $E_n(T, \mathbf{0}) \geq \mathcal{E}(T)\, n$ as required. □

5. Explosion formula

We introduce a recursive formula for the escape numbers $E_n(T, r)$, from which we will deduce Theorem 2(i) and many other results. It will be convenient to work with the indicator $e_n = e_n(T, r) := \mathbb{1}[\text{particle } n \text{ escapes to infinity}]$, so that $E_n = \sum_{i=1}^n e_i$. Let $e(T, r)$ denote the binary **escape sequence** $(e_1, e_2, \ldots)$. We sometimes abbreviate a sequence $(a_1, a_2, \ldots)$ to $a_1 a_2 \cdots$.

For a tree T, let ι be the unique child of the root o, and call ι the **base** of the tree. For any vertex $v \neq o$ of T, the **subtree T_v based at** v is defined to be the subgraph of T induced by the parent $v^{(0)}$ of v together with v and all its descendants. The subtree T_v has root $v^{(0)}$ and base v. For a rotor configuration r on T, we write r_v for its restriction to T_v (more precisely, to the non-root vertices of T_v). If ι has children $\iota^{(1)}, \ldots, \iota^{(b)}$, we abbreviate $T_{\iota^{(i)}}$ to T_i and $r_{\iota^{(i)}}$ to r_i, and call $T_1, \ldots, T_b$ the **principal branches** of T.

Similarly if T is a finite tree with a set of sinks S, as in Section 3, we write $S_v := S \cap V(T_v)$, and $S_i := S_{\iota^{(i)}}$. We also write $e_n(T, S, r) := \mathbb{1}[\text{particle } n \text{ escapes to } S]$, so again $E_n = \sum_{i=1}^n e_i$.

Let $\mathbb{N}_+ = \{1, 2 \ldots\}$. For sequences in $\mathbb{N}^{\mathbb{N}_+}$ we denote addition by $(a_1, a_2, \ldots) + (b_1, b_2, \ldots) := (a_1 + b_1, a_2 + b_2, \ldots)$, and define the **shift** operator θ by
$$\theta(a_1, a_2, \ldots) := (0, a_1, a_2, \ldots)$$
and the **explosion** operator $\mathfrak{X}$ by
$$\mathfrak{X}(a_1, a_2, \ldots) := (1^{a_1}, 0, 1^{a_2}, 0, \ldots)$$
where 1^k denotes a string of k 1s. We define the **majorization** order $\preceq$ on sequences by $(a_1, a_2, \ldots) \preceq (b_1, b_2, \ldots)$ if and only if $\sum_1^n a_i \leq \sum_1^n b_i$ for all n.

Theorem 11 (Explosion formula)**.**

(i) *Let T be a tree with base ι and principal branches $T_1, \ldots, T_b$, and fix a rotor configuration r. Then*
$$e(T, r) = \mathfrak{X}\left(\sum_{i=1}^{r(\iota)} \theta e(T_i, r_i) + \sum_{i=r(\iota)+1}^{b} e(T_i, r_i) \right). \tag{3}$$

(ii) *If T is a finite tree with a set of sinks S,* (3) *holds similarly for $e(T, S, r)$ and $e(T_i, S_i, r_i)$.*

(iii) *In the case of an infinite tree as in (i), the collection of escape sequences* $(e(T_v, r_v))_{v \in V_o}$ *is the unique maximal* $\{0,1\}^{\mathbb{N}_+}$*-valued solution to the system of equations obtained by applying* (3) *to each of the subtrees* $(T_v)_{v \in V_o}$ *(in place of* T*). Here maximality means that if* $(e'(v))_{v \in V_o}$ *is another solution, then* $e(T_v, r_v) \succeq e'(v)$ *for all* v*.*

We remark that the system of equations in (iii) above does not generally have a unique solution; for example $e(T_v, r_v) \equiv (0, 0, \ldots)$ is always a solution. It is not *a priori* obvious that the system of equations has a maximal solution in the sense asserted in (iii) (as opposed to the weaker sense of being majorized by no other solution).

We will use Theorem 11 to prove Theorem 2(ii) and Theorem 5, as well as the following propositions, of which the first two will be used later, and all are of independent interest.

Proposition 12 (Finite modification)**.** *If* r *and* r' *are rotor configurations differing at only finitely many vertices, then for all* n*,*

$$|E_n(T, r) - E_n(T, r')| \leq \sum_{v \in V_o} |r(v) - r'(v)|.$$

Proposition 13 (Subtrees)**.** *If the tree* T' *is a subgraph of the tree* T*, with the same root, then* $e(T', \mathbf{0}) \preceq e(T, \mathbf{0})$ *(i.e. for all* n *we have* $E_n(T', \mathbf{0}) \leq E_n(T, \mathbf{0})$*).*

Proposition 14 (Rotor orders)**.** *If* T, T' *have the same underlying graph, but different cyclic orders of neighbours, and* r *is a rotor configuration satisfying* $r(v) \in \{0, b(v)\}$ *for each* v *(such as* $\mathbf{0}$*), then* $E_n(T, r) = E_n(T', r)$ *for all* n*.*

Proposition 15 (Non-uniqueness)**.** *There exist uncountably many distinct trees with the same escape sequence for initial configuration* $\mathbf{0}$*.*

For sequences a, b we write $a \prec b$ if $a \preceq b$ but $a \neq b$.

Proposition 16 (Self-majorization)**.** *Let* T *be any infinite tree and let* $e = e(T, \mathbf{0})$*. For any* $n > 0$ *we have the strict majorization* $e \succ (e_{n+1}, e_{n+2}, \ldots)$*.*

The above proposition implies in particular that $e(T, \mathbf{0})$ cannot be a periodic sequence. Periodicity is possible for other rotor configurations (for example the constant configuration $\mathbf{1}$ on the binary tree has $e = 101010\cdots$, as noted in [17]). Not every sequence e satisfying the self-majorization condition of Proposition 16 is equal to $e(T, \mathbf{0})$ for some tree T. For example, Theorem 11 and Proposition 16 may be used to check that no sequence starting $110010101010\cdots (= \mathfrak{X}(201111\cdots))$

can arise in this way. It is an open problem to characterize the set of all possible sequences $e(T, \mathbf{0})$ as T varies.

Now we turn to the proofs.

Proof of Theorem 11. We suppress the rotor configuration in the notation $e(T_v, r_v)$ for escape sequences, since it will always be the appropriate restriction.

Part (i) is an elementary consequence of the behaviour of the rotor at ι. Suppose first that $r(\iota) = 0$, so that ι will send the particle to each of its children before returning it to o. When it is sent to the ith child, either the particle escapes (and then is returned to ι via o), or it is returned to ι by the child, according to the value of $e_1(T_i)$. In either case, the rotor at ι is then incremented and we proceed to the next child. Finally, the rotor points to o, and we get a return to o. Thus $e(T)$ starts with the sequence 1^k0, where $k = \sum_{i=1}^b e_1(T_i)$. At this point the rotor is in its initial position, and each principal branch has received a particle once from ι, so the process repeats using $e_2(T_i)$, and so on, and we deduce $e(T) = \mathfrak{X} \sum_{i=1}^b e(T_i)$ as required in this case. The case of general $r(\iota)$ is similar, except that we must first apply the shift θ to the sequences $e(T_i)$ of those branches to which the rotor will not point before pointing to o, because they do not contribute any escapes to the first block of 1s of $e(T)$.

An identical argument to the above gives the finite case (ii). To prove (iii), consider the truncated system (T^h, S^h, r^h) (as defined in Section 3). Applying (ii) to every subtree yields a system of equations giving the escape sequence of each subtree in terms of those of its principal branches. Together with the boundary condition that $e(T^h_s) = 111\cdots$ for every $s \in S^h$, these equations determine all the escape sequences. Note that $\mathfrak{X}$ is an increasing function with respect to the order $\preceq$ (on both the domain and the range). Therefore if e' is another $\{0,1\}^{\mathbb{N}_+}$-valued solution to the system of equations, proceeding iteratively starting from the sinks, we deduce that $e'(v) \preceq e(T^h_v)$ for each subtree. Finally we apply Lemma 9 to deduce that the same conclusion holds for the infinite tree. □

Proof of Theorem 2(i). For a finite tree with sinks, the required monotonicity follows from the observation that the right side of (3) is decreasing in $r(\iota)$ and increasing in $e(T_i)$, where sequences are ordered by majorization. To deduce the infinite case, apply Lemma 9. □

Proof of Proposition 12. The main observation, which is straightforward to check, is that if a, a' is a pair of integer sequences satisfying

$\sum_{i=1}^{n}(a_i - a_i') \in \{0,1\}$ for all n, then the pair $\mathfrak{X}(a), \mathfrak{X}(a')$ satisfies the same condition.

Suppose first that r and r' differ only at a single vertex v, where $r'(v) = r(v) + 1$. In this case the explosion formula (3) applied to T_v gives $e(T_v, r_v) = \mathfrak{X}(a + e)$ and $e(T_v, r_v') = \mathfrak{X}(a + \theta e)$ for a non-negative sequence a and a binary sequence e. Hence by the observation above, $E_n(T_v, r_v) - E_n(T_v, r_v') \in \{0,1\}$ for all n. If the last statement holds for some vertex u (in place of v), then by the above observation again, it also holds for its parent $u^{(0)}$. Applying this iteratively shows that it holds for ι.

The required inequality now follows by repeatedly applying the argument above, since we may move from r to r' via a sequence of configurations, in which one rotor is incremented or decremented at each step. □

Proof of Proposition 13. Let r be a rotor configuration on T given by

$$r(v) = \begin{cases} 0, & v \in V(T'); \\ b(v), & v \notin V(T'). \end{cases}$$

In the rotor walk on T with initial configuration r, whenever the particle enters a subtree that is not present in T', it is eventually returns to the parent without escaping, by Theorem 2(iii). Therefore $e(T, r) = e(T', \mathbf{0})$. The result now follows by Theorem 2(i). □

Proof of Proposition 14. If $r(\iota) \in \{0, b(\iota)\}$, the right side of (3) is unchanged by reordering the principal branches. Iterate to get the result for finite trees, and apply Lemma 9 for the infinite case. □

Proof of Proposition 15. Let $a = (a_1, a_2, \ldots)$ be a strictly increasing sequence of non-negative integers, and let $T(a)$ be a 'thinned comb' – a tree consisting of a singly-infinite path $(o, x_0, x_1, \ldots)$, together with additional singly-infinite paths attached at each of the vertices $\{x_{a_i}\}_{i \geq 1}$. It is straightforward to check that

$$e(T(a), \mathbf{0}) = 1\,0^{a_1}\,1\,0^{a_2}\,1\,\cdots.$$

Thus there exist uncountably many pairs of such sequences (a, b) with the property that

$$e(T(a), \mathbf{0}) + e(T(b), \mathbf{0}) = 2\;1\;0^1\;1\;0^3\;1\;0^7\;1\;\cdots\;0^{2^i-1}\;1\cdots$$

since we may choose arbitrarily which 1s come from a and which from b. For any such pair we may form a tree by joining $T(a)$ and $T(b)$ together at their roots, and adding a new edge from this vertex to a new vertex, which we designate the new root. The explosion formula shows that the resulting trees all give the same escape sequence. □

Proof of Proposition 16. As remarked in the introduction, we may assume without loss of generality that every vertex has at least one child, i.e. every subtree is infinite.

For the weak inequality $e \succeq (e_{n+1}, e_{n+2}, \ldots)$, note that the rotor configuration r after n particles have been added obviously satisfies $r(v) \geq \mathbf{0}(v)$ for all $v \in V_o$, and apply Theorem 2(i).

It remains to prove $e \neq (e_{n+1}, e_{n+2}, \ldots)$. This indeed holds for $n = 1$, because equality would imply that e is a constant sequence, which is impossible since the first particle escapes but some do not. For a general $n > 0$, consider the configuration r after n rotor particles have been run, and let v be some vertex that has been visited exactly once at that time (such a vertex exists by the proof of Proposition 8). From the $n = 1$ case just considered, the subtree T_v based at v has the property that its subsequent escape sequence is strictly majorized by the initial sequence, $e(T_v, r_v) \prec e(T_v, \mathbf{0}_v)$. On the other hand, for *every* vertex u we have the weak inequality $e(T_u, r_u) \preceq e(T_u, \mathbf{0}_u)$. The right side of the explosion formula (3) is strictly monotone in the sense that: if for two rotor configurations r, s we have $r(\iota) \geq s(\iota)$, while for each principal branch we have $e(T_i, r_i) \preceq e(T_i, s_i)$, with one of these majorizations being strict, then $e(T, r) \prec e(T, s)$. Applying this iteratively to the sequence of ancestors of v gives the result. $\square$

6. Constant configurations on regular trees

Theorem 5 on regular trees is a consequence of the following result. Fix $b \geq 2$, and for a positive integer t, write z_t for the number of trailing zeros in its base-b expansion and ℓ_t for the last nonzero digit, so $t \equiv \ell_t b^{z_t} \pmod{b^{z_t+1}}$ with $\ell_t \in \{1, \ldots, b-1\}$.

Proposition 17 (Constant rotor configurations)**.** *Let* $\mathbf{k}$ *be the constant rotor configuration on the* b*-ary tree* $\mathbb{T}_b$ *given by* $\mathbf{k}(v) = k$ *for all* v*. The escape sequence is given as follows.*

$$e(\mathbb{T}_b, \mathbf{k}) = \begin{cases} 1\,0^{z_1}\;1\,0^{z_2}\;1\,0^{z_3}\cdots & \textit{if } k = 0; \\ 1\,0^{\mathbb{1}[\ell_1 = b-k]}\;1\,0^{\mathbb{1}[\ell_2 = b-k]}\;1\,0^{\mathbb{1}[\ell_3 = b-k]}\cdots & \textit{if } 0 < k < b. \end{cases}$$

Recall that $e(\mathbb{T}_b, \mathbf{b}) = 000\cdots$ by Theorem 2(iii). It is interesting that $e(\mathbb{T}_b, \mathbf{k})$ contains consecutive zeros when $k = 0$, but not when $0 < k < b$.

Proof of Proposition 17. By Theorem 11(i), the escape sequence $e = e(\mathbb{T}_b, \mathbf{k})$ satisfies

$$e = \mathfrak{X}\big(k\,\theta e + (b-k)\,e\big). \tag{4}$$

Also, for $k < b$ it is clear that the first particle escapes, i.e. $e_1 = 1$. It turns out that these facts are sufficient to determine e (we do not need to make explicit use of the maximality in Theorem 11(iii)).

For the case $k = 0$, we note that if e is any sequence starting with

$$1\,0^{a_1}\,1\,0^{a_2}\,1\;\cdots\;0^{a_m}\,1,$$

then $\mathfrak{X}\big(b\,e\big)$ starts with

$$1^b\,0^{a_1+1}\,1^b\,0^{a_2+1}\,1^b\;\cdots\;0^{a_m+1}\,1^b.$$

Therefore, we can obtain the sequence $e(\mathbb{T}_b, \mathbf{0})$ by starting with the singleton sequence 1 and repeatedly applying the above transformation. Each finite sequence that results is necessarily a prefix of the next, and of the desired infinite sequence. Since $1^r = 1\,0^0\,1\,0^0 \cdots 1$, it is easy to deduce that the claimed expression for $e(\mathbb{T}_b, \mathbf{0})$ holds.

The case $0 < k < b$ is similar but a little more complicated. Write $X := 1\,0$, and suppose e is any sequence starting with a concatenation of the form

$$Y_1\,Y_2\,\cdots\,Y_m,$$

where each Y_i is either X or 1. Then it is straightforward to check that substituting e into the right side of (4) gives a sequence starting

$$W\,Y_1\,W\,Y_2\,\cdots\,W\,Y_m,$$

where $W := 1^{b-k-1}\,X\,1^{k-1}$. Now the claimed expression follows by repeatedly applying this starting from the sequence 1. $\square$

Proof of Theorem 5. Note first that $\mathcal{E}(\mathbb{T}_b) = (b-1)/b$ (by (1), for example). The required bounds on $E_n(\mathbb{T}_b, \mathbf{k})$ follow from Proposition 17 by routine computations, which we summarize below.

Observe that restricting to positions n just before a 1 in the escape sequence (i.e. such that $e_{n+1}(\mathbb{T}_b, \mathbf{k}) = 1$) does not change the lim inf or lim sup of Δ_n. Let D_m denote the sum of the base-b digits of m.

For the case $k = 0$, we have

$$S_m := \sum_{i=1}^{m} z_i = \frac{m - D_m}{b-1}.$$

Therefore if $n+1$ is the position of the $(m+1)$st 1 in $e(\mathbb{T}_b, \mathbf{0})$, then

$$E_n = m; \qquad n = S_m + m; \qquad E_n - \mathcal{E}n = \frac{D_m}{b}.$$

We deduce that for any integer $s \geq 0$, any $b^{s-1} \leq m < b^s$, and n as above,

$$\frac{1}{b} \leq E_n - \mathcal{E}n \leq \frac{b-1}{b}\,s$$

where the lower and upper bounds are attained by taking $m = b^{s-1}$ and $m = b^s - 1$ respectively. For m in this range we have $\log_b n = \log_b(m + S_m) = s + O(1)$ as $m \to \infty$. The claimed expressions for $\liminf_{n\to\infty} \Delta_n$ and $\limsup_{n\to\infty} \Delta_n$ follow.

Now consider the case $k \geq 1$. We have

$$\sum_{i=1}^{m} \mathbb{1}[\ell_i = b - k] = \frac{m - D_m}{b - 1} + \sum_{j=0}^{\infty} \mathbb{1}[m_j \geq b - k],$$

where $\cdots m_2\, m_1\, m_0$ is the base-b expansion of m. Similarly to the previous case, we deduce that

$$-\frac{k-1}{b}\, s \leq E_n - \mathcal{E}n \leq \frac{b-k-1}{b}\, s,$$

for n satisfying $E_n = m$ where m has s base-b digits, with the bounds being attained when m has all digits $b - k$, and all digits $b - k - 1$, respectively. The claimed expressions follow. □

7. Brushes

In preparation for the proof of Theorem 4, we next provide a family of recurrent trees (which are interesting in their own right) with $E_n(T, \mathbf{0})$ of order $n^{1-\epsilon}$, for arbitrary $\epsilon > 0$.

The d**-brush** $\mathbb{B}^d$ is a tree defined as follows. The 1-brush is a singly infinite path. For $d > 1$, the d-brush is a singly infinite path with a $(d-1)$-brush attached, via its root, at each non-root vertex. Note that a d-brush has two principal branches: a d-brush and a $(d-1)$-brush. (Another interpretation is as follows. Index the non-root vertices of $\mathbb{T}_2$ by binary sequences, with the base having the empty sequence and the children of a vertex having the vertex's label with 0 or 1 appended. With this labeling, the d-brush is a subgraph of $\mathbb{T}_2$ that consists of all vertices with Hamming weight less than d. It is also easy to embed $\mathbb{B}^d$ in $\mathbb{Z}^d$.) Since we will be concerned only with the configuration $\mathbf{0}$, the orders of children do not matter by Proposition 14.

Proposition 18 (Brushes)**.** *The d-brush has $E_n(\mathbb{B}^d, \mathbf{0}) \sim cn^\beta$ as $n \to \infty$, for $\beta = 1 - 2^{1-d}$ and some $c = c(d) \in (0, \infty)$.*

The proof uses the following lemmas. The constant $c(d)$ is easily computable (recursively) from the lemmas, but this will not be needed.

Lemma 19. *Let $E_n^d = E_n(\mathbb{B}^d, \mathbf{0})$ be the escape number for the d-brush. For fixed $d \geq 2$, define the sequence $(s_i) = (s_i^d)_{i\geq 1}$ by $s_1 = 1$ and*

$$s_{i+1} - s_i = E^{d-1}_{s_1+\cdots+s_i}, \quad i \geq 1.$$

Then

$$E^d_{s_1+\cdots+s_i} = s_i, \quad i \geq 1. \tag{5}$$

For example, since $E^1_n = 1$ for all $n > 0$, we have $s^2_i = i$, so (5) gives $E^2_{i(i+1)/2} = i$.

Proof. Fix $d \geq 2$. Writing $e^d = e(\mathbb{B}^d, \mathbf{0})$, by the explosion formula, Theorem 11(i), we have

$$e^d = \mathfrak{X}(e^d + e^{d-1}),$$

and therefore for all $n \geq 1$,

$$E^d_{E^d_n+E^{d-1}_n+n} = E^d_n + E^{d-1}_n. \tag{6}$$

We prove (5) by induction on i. It holds for $i = 1$ since the first particle escapes. Assuming it holds for i, and taking $n = s_1 + \cdots + s_i$, we have

$$E^d_n + E^{d-1}_n = s_i + (s_{i+1} - s_i) = s_{i+1},$$

so equation (6) becomes

$$E^d_{n+s_{i+1}} = s_{i+1}. \qquad \square$$

Lemma 20. *Suppose $f : \mathbb{N} \to \mathbb{N}$ satisfies $f(n) \sim an^\alpha$ as $n \to \infty$, with $a \in (0, \infty)$ and $\alpha \in (0, 1)$. Let $s_1 = 1$, and inductively $s_{m+1} = s_m + f(s_1 + \cdots + s_m)$ for $m \geq 1$. If $g : \mathbb{N} \to \mathbb{N}$ is increasing and satisfies $g(s_1 + \cdots + s_m) = s_m$ for all m, then $g(n) \sim bn^\beta$, where $\beta = (1 + \alpha)/2$, and $b = \sqrt{\frac{2a}{1+\alpha}}\left(\frac{1-\alpha}{2}\right)^{1+\alpha}$.*

Proof. Write $S_m = \sum_{i=1}^m s_i$. We will show that $s_m \sim dm^\gamma$ with

$$\gamma = \frac{1 + \alpha}{1 - \alpha} \qquad \text{and} \qquad d = \left(\frac{a(1 - \alpha)^{1+\alpha}}{2^\alpha(1 + \alpha)}\right)^{1/(1-\alpha)}.$$

This implies $S_m \sim \frac{d}{1+\gamma} m^{1+\gamma}$, and the lemma then follows by the given properties of g, since $\gamma/(1 + \gamma) = \beta$.

Let $L = \limsup_{m\to\infty} s_m/m^\gamma$. We first show $L < \infty$. Since $S_m \to \infty$, for some m_0, for all $m \geq m_0$ we have $f(S_m) < 2aS^\alpha_m$. Let $C \geq \max_{m\leq m_0} s_m/m^\gamma$ be such that for all m

$$Cm^\gamma + C^\alpha \frac{2a}{(1 + \gamma)^\alpha}(m + 1)^{\gamma-1} < C(m + 1)^\gamma$$

(it is easy to see such C exists since $\alpha < 1$). We now prove by induction that $s_m < Cm^\gamma$ for all m (hence $L \leq C$). This is known for $m \leq m_0$.

If it holds up to m, then $S_m \le \frac{C}{1+\gamma}(m+1)^{1+\gamma}$, so

$$\begin{aligned} s_{m+1} = s_m + f(S_m) &\le Cm^\gamma + 2a\left(\frac{C}{1+\gamma}\right)^\alpha (m+1)^{\alpha(1+\gamma)} \\ &= Cm^\gamma + C^\alpha \frac{2a}{(1+\gamma)^\alpha}(m+1)^{\gamma-1} \\ &< C(m+1)^\gamma, \end{aligned}$$

completing the induction.

In a similar manner, integrating, using $f(n) \sim an^\alpha$, and integrating again gives the sequence of inequalities

$$\begin{aligned} \limsup \frac{S_m}{m^{1+\gamma}} &\le \frac{L}{\gamma+1}; \\ \limsup \frac{s_{m+1}-s_m}{m^{\gamma-1}} &\le a\left(\frac{L}{\gamma+1}\right)^\alpha; \\ L = \limsup \frac{s_m}{m^\gamma} &\le \frac{a}{\gamma}\left(\frac{L}{\gamma+1}\right)^\alpha. \end{aligned}$$

Since $\alpha < 1$, solving for L gives the upper bound

$$L \le \left(\frac{a}{\gamma(\gamma+1)^\alpha}\right)^{1/(1-\alpha)}. \tag{7}$$

Similarly, let $\ell = \liminf_{m\to\infty} s_m/m^\gamma$. We first show $\ell > 0$. For $m > m_1$ we have $f(S_m) > aS_m^\alpha/2$. Take $0 < c < \min_{m \le m_0} s_m/m^\gamma$, small enough that for all m

$$cm^\gamma + c^\alpha \frac{a}{2(1+\gamma)^\alpha} m^{\gamma-1} > (m+1)^\gamma.$$

Then $S_m \ge \frac{c}{1+\gamma} m^{\gamma-1}$, and so

$$\begin{aligned} s_{m+1} = s_m + f(S_m) &\ge cm^\gamma + \frac{a}{2}\left(\frac{c}{1+\gamma}\right)^\alpha m^{\gamma-1} \\ &= cm^\gamma + c^\alpha \frac{a}{2(1+\gamma)^\alpha} m^{\gamma-1} \\ &> c(m+1)^\gamma. \end{aligned}$$

By induction we find $s_m > cm^\gamma$ for all m.

By the same argument as for the lim sup, but with the inequalities reversed, this implies

$$\ell \ge \frac{a}{\gamma}\left(\frac{\ell}{\gamma+1}\right)^\alpha.$$

Solving shows that ℓ is bounded below by the same quantity as in (7), hence $\ell = L$. $\square$

Proof of Proposition 18. For the 1-brush we have $E_n(\mathbb{B}^1, \mathbf{0}) = 1$ for all n. For other d the result follows by induction from Lemmas 19 and 20. □

8. Large-discrepancy trees

In this section we prove Theorem 4. We begin with a corollary of Proposition 18.

Corollary 21. *For any $\epsilon > 0$ there exists a tree T of maximum degree 3 which is recurrent for random walk and satisfies $E_n(T, \mathbf{0}) > n^{1-\epsilon}/2$.*

Proof. For $k, d \geq 1$, let $T(k,d)$ be a tree defined as follows: start with $(\mathbb{T}_2)^k$, a binary tree of depth k, and attach 2 disjoint copies of the brush $\mathbb{B}^d$, via their roots, to each of its 2^{k-1} leaves. This is recurrent since it has only countably many ends. Take d sufficiently large that $2^{1-d} < \epsilon$. We claim that $T(k,d)$ satisfies the claimed bound for k large enough. In what follows we suppress the rotor configuration in the notation E_n, since it will always be $\mathbf{0}$.

First note that if H is any tree satisfying $E_n(H) \sim an^\alpha$, with $\alpha < 1$, then the tree H' having two disjoint copies of H as its principal branches satisfies $E_n(H') \sim 2an^\alpha$, by Theorem 11(i). Using Proposition 18 and applying this repeatedly shows that for any d, for k sufficiently large we have $E_n(T(k,d)) \sim n^\beta$ with $\beta = 1 - 2^{1-d}$, and so for n sufficiently large (say $n > N = N(k(d))$), we have $E_n(T(k,d)) \geq n^\beta/2$.

Now for any k' consider $E_n(T(k',1))$ (which is a finite binary tree with a singly infinite path attached at each leaf). We claim that $E_n(T(k',1)) \geq n/2 \geq n^\beta/2$ for all $n \leq 2^{k'}$. Indeed, the escapes in $T(k',1)$ occur at precisely the times of the first $2^{k'}$ escapes in the infinite binary tree $\mathbb{T}_2$, as may be seen from the proof of Proposition 17 (case $k = 0$), for example. And the claimed inequality then follows from Theorem 2(ii).

Now take k' large enough that $2^{k'} > N$ and $k' \geq k$, and observe that $T(k',d)$ contains (isomorphic copies of) both $T(k,d)$ and $T(k',1)$ as subgraphs with the same root. Therefore by Proposition 13, for all n,

$$E_n(T(k',d)) \geq \max\Big[E_n(T(k,d)), E_n(T(k',1))\Big] \geq n^\beta/2. \qquad \square$$

Proof of Theorem 4. Let $(\epsilon_k)_{k\geq 1}$ be a $(0,1]$-valued sequence to be determined later. Construct a tree T by taking an infinite path with vertices $o, x_1, x_2, \ldots$, and attaching a recurrent tree $T(k)$ that satisfies $E_n(T(k), \mathbf{0}) > n^{1-\epsilon_k}/2 \ \forall n$, via its root, to vertex x_k, for each $k \geq 1$. Such trees $T(k)$ exist by Corollary 21, and the resulting tree T is clearly recurrent. We will choose the ϵ_k so that it satisfies the required bound. In the following the rotor configuration is always $\mathbf{0}$.

First, if H is any infinite tree, let $H^{(k)}$ be the tree formed by attaching a path of k edges to the root of H, and taking the other end of this path as the new root. If $e(H) = 1\,0^{a_1}\,1\,0^{a_2}\,1\cdots$ then by Theorem 11, $e(H^{(k)}) = 1\,0^{a_1+k}\,1\,0^{a_2+k}\,1\cdots \preceq 1\,0^{ka_1+k}\,1\,0^{ka_2+k}\,1\cdots$, so we have $E_n(H^{(k)}) \geq E_{\lceil n/k\rceil}(H)$ for all n.

Since the tree T constructed in the first paragraph contains each $T(k)^{(k)}$ as a subgraph, by Proposition 13 we deduce that it satisfies

$$E_n(T) \geq \frac{1}{2}\left(\frac{n}{k}\right)^{1-\epsilon_k} \geq \frac{n}{2kn^{\epsilon_k}} \tag{8}$$

for all $n, k \geq 1$.

Now suppose we are given f with $f(n) = o(n)$. Since the desired statement involves only sufficiently large n we may assume without loss of generality that $f(n) < n/4$ for all n. For each n, let $k(n) = \left\lfloor \frac{1}{2}\sqrt{n/f(n)}\right\rfloor$, and note that this is at least 1 by the last assumption. Let

$$\epsilon_k := 1 \wedge \min\Big\{\log_n \sqrt{n/f(n)} : k(n) = k\Big\}.$$

Since $k(n) \to \infty$ as $n \to \infty$, this is the minimum of a finite set, and $f(n) < n$, so $\epsilon_k > 0$. This choice ensures that for all n we have $n^{\epsilon_{k(n)}} \leq \sqrt{n/f(n)}$, therefore taking $k = k(n)$ in (8) gives $E_n(T) \geq f(n)$ for all n. $\square$

9. Random configurations

In this section we prove Theorem 6(i). We begin with an exact result on finite regular trees, which is somewhat remarkable in that it holds regardless of the initial rotor configuration.

Proposition 22 (Finite regular trees). *Consider $\mathbb{T}_b^{h+1}$, the b-ary tree truncated at level $h+1$, with an arbitrary initial rotor configuration. Let S be the set of the b^h leaves at level $h+1$. If $1 + b + \cdots + b^h$ rotor particles are started in succession at o, and stopped when they enter $S \cup \{o\}$, then exactly one particle is absorbed at each vertex of S, and the rest are absorbed at o.*

Proof. Write $t(h) = 1 + b + \cdots + b^h$. We use induction on h. The result clearly holds when $h = 1$. Assume it holds for $h - 1$. Let r be the initial rotor configuration.

We first apply the explosion formula to bound below the total number of particles escaping to S. By the inductive hypothesis, each of the b principal branches T_i has $E_{t(h-1)}(T_i, S_i, r_i) = b^{h-1}$. Therefore, in the worst case $r(\iota) = b$, the sequence $z := \sum_{i=1}^{b} \theta e(T_i)$ satisfies $\sum_{j=1}^{1+t(h-1)} z_j = b^h$. By Theorem 11(ii), this implies $E_n(T) \geq b^h$ for $n = 1+t(h-1)+b^h-1 = t(h)$, i.e. at least b^h of the $t(h)$ particles are absorbed by S.

Now suppose that some leaf $w \in S$ absorbs no particles. Then by the inductive hypothesis, the principal branch containing w must have received a particle from ι strictly fewer than $t(h-1)$ times, hence (by considering the rotor at ι), no principal branch received a particle strictly more than $t(h-1)$ times. Therefore by the inductive hypothesis again, no leaf of S absorbed more than one particle, which contradicts the above bound on the total absorptions by S. □

Corollary 23. *Let R be an i.i.d. random rotor configuration on $\mathbb{T}_b$ satisfying $\mathbb{E}R(v) < b-1$. There exist $\delta, c, C > 0$ such that for all n,*

$$\mathbb{P}\big(E_n(\mathbb{T}_b, R) < \delta n\big) \leq Ce^{-cn}.$$

The proof of the above result will use a version of the Abelian property. This requires the concept of simultaneous rotor walks, which we describe next (see e.g. [12] for more details). We will give the property we need for finite trees, and then use Lemma 9. (An alternative approach would be to formulate a version of the Abelian property for infinite graphs.)

Let T be a finite tree with root o, and $S \not\ni o$ a non-empty set of its leaves, as in Section 3. At each vertex there is a rotor, as usual, and at each vertex there is some non-negative integer number of particles. At each step of the process, we choose a vertex $v \notin S \cup \{o\}$ at which there is at least one particle (if such exists), and **fire** v; that is, increment the rotor at v, and move one particle from v in the new rotor direction.

Lemma 24 (Abelian property)**.** *Let T be a finite tree with set of sinks S and initial rotor configuration r. If we start with n particles at the base ι and perform any legal sequence of firings until all particles are in $S \cup \{o\}$, then the process terminates in a finite number of steps, and the number of particles in S is exactly $E_n(T, S, r)$.*

We remark that even stronger statements hold: the number of times each edge is traversed, and in particular the final distribution of the particles in S and the final rotor configuration also do not depend on the sequence of firings.

Proof. We will adapt the Abelian property as formulated in [12], for which we need to modify the graph slightly. Assume $\iota \notin S$ (otherwise the result is trivial). For each edge of T not incident to $S \cup \{o\}$, replace it with two directed edges, one in each direction. For each edge incident to $S\cup\{o\}$, replace it with a single edge directed towards $S\cup\{o\}$. Finally, add an extra 'global sink' vertex Δ and add a directed edge from each vertex in $S \cup \{o\}$ to Δ. By [12, Lemma 3.9], starting from a given configuration of particles and rotors, in any legal sequence of firings on this directed graph (stopping when all particles are at Δ), each vertex fires the same number of times. In particular, if n particles start at ι, this means that the number of particles that reach S does not depend on the firing sequence (because it is the number of firings in S). For the firing sequence in which we move only one particle until it reaches Δ, then move the next particle, and so on, this number is $E_n(T, S, r)$. (The initial transitions from o to ι in our usual formulation of the rotor walk clearly do not matter). □

Proof of Corollary 23. It suffices to prove the claimed result with n of the form $1+b+\cdots+b^h$ (for h an integer), since this implies the bound (with different constants) for all n.

Since $\mathbb{E}R(v) < b-1$, the branching process from the proof of Theorem 6(ii) is supercritical, and hence with positive probability there is a live path starting at ι, implying that the first particle escapes. Thus

$$p := \mathbb{P}(E_1(\mathbb{T}_b, R) = 1) > 0.$$

Let X be the set of vertices v at level $h+1$ for which the subtree T_v based at v satisfies $E_1(T_v, R_v) = 1$. We claim that for $n := 1+b+\cdots+b^h$ we have

$$E_n(\mathbb{T}_b, R) \geq \#X. \tag{9}$$

Once this is proved, the result follows from a standard large-deviation bound (e.g. [9, Theorem 9.5]), since $\#X$ has Binomial(b^h, p) distribution.

To prove (9), by Lemma 9 it suffices to prove that for all $H > h$,

$$E_n(\mathbb{T}_b^H, S^H, R^H) \geq \#X. \tag{10}$$

In the finite tree $\mathbb{T}_b^H$, start n particles at ι, stopping them when they enter $S^{h+1} \cup \{o\}$. By Proposition 22, this results in one particle at each vertex of S^{h+1} and the rest at o, and the rotors at distance $h+1$ or greater from o are not affected. Now for a vertex $v \in X$, continue the rotor walk on $\mathbb{T}_b^H$ for the particle located there until it enters $S^H \cup \{o\}$; since in the infinite tree the particle would escape without leaving the subtree T_v, the same applies in the truncated tree. Therefore, allowing each of the particles at elements of X to continue in this way results in all of them reaching S^H. Finally we can continue the rotor walks for the remaining particles (those at the other vertices of S^{h+1}) until they reach $S^H \cup \{o\}$. By Lemma 24, (10) follows. □

Lemma 25 (Liminf recursion)**.** *Fix any tree T and rotor configuration r, and let $T_1, \ldots, T_b$ be the principal branches. Let*

$$\ell := \liminf_{n\to\infty} \frac{E_n(T,r)}{n}; \quad \ell_i := \liminf_{n\to\infty} \frac{E_n(T_i,r_i)}{n}, \quad i = 1, \ldots, b;$$

then

$$\ell \geq 1 - \frac{1}{1+\sum_{i=1}^b \ell_i}.$$

Proof. Write $E_n = E_n(T,r)$ and $E_n^i = E_n(T_i, r_i)$. An application of the explosion formula, Theorem 11(i), shows that $E_{m+n} = m$ for some $m = m_n = c_n + \sum_{i=1}^b E_n^i$ with $c_n \in [0,b]$ (the error term c_n comes from the shifts θ in the explosion formula). Hence

$$\frac{E_{m+n}}{m+n} = 1 - \frac{1}{1+m/n} = 1 - \frac{1}{1+c_n/n+\sum_i (E_n^i/n)}.$$

Since $\liminf_n \sum_i (E_n^i/n) \geq \sum_i \ell_i$, and E_n is monotone in n, the result follows. □

Proof of Theorem 6(i). Let $L = \liminf_{n\to\infty} E_n(\mathbb{T}_b, R)/n$. By Corollary 23 and the Borel-Cantelli lemma we have $L \geq \delta$ a.s., where $\delta > 0$. On the other hand, by Lemma 25, if for some constant $a > 0$ we have $L \geq a$ a.s., then $L \geq 1 - 1/(1+ba)$ a.s. Applying this repeatedly gives $L \geq 1 - b^{-1} = \mathcal{E}$ a.s., since this is the fixed point of the iteration $a \mapsto 1 - 1/(1+ba)$. Now use Theorem 1. □

10. Rotors about to point away from the root

In this section we prove Theorem 3. We say that a rotor configuration r is ***b*-free** if for every $v \in V_o$ we have $r(v) \neq b(v)$; i.e. every rotor will send its next particle away from the root. We note in particular that this implies $b(v) \geq 1$ for every v, and hence every subtree T_v is infinite. (In any case, trees in which some vertices have no children are of little interest to us, as remarked in the introduction).

Lemma 26. *For any b-free rotor configuration r on a tree T, we have for all $n \geq 1$,*

$$E_n(T,r) \geq \mathcal{E}n/2.$$

Proof. We proceed by induction on n. It is easy to see that the first particle escapes, so the inequality holds for $n = 1$. Now take $n > 1$ and assume that it holds for all smaller n, for all T and b-free r.

Let $T_1, \ldots, T_b$ be the principal branches of T, and write $\mathcal{E}_i = \mathcal{E}(T_i)$ and $\mathcal{E} = \mathcal{E}(T)$. Let n_i be the total number of particles that enter T_i

when n rotor walks are run from o. Since $n-E_n(T)$ is the number that return to o, we have

$$n_i \geq n - E_n(T) - 1.$$

Also, since $n > 1$ we have $n_i < n$, so by the inductive hypothesis,

$$E_{n_i}(T_i) \geq \mathcal{E}_i n_i/2.$$

Note that $\sum_{i=1}^b \mathcal{E}_i = \mathcal{E}/(1-\mathcal{E})$ (similarly to (1)). Hence, using the above inequalities, we have

$$E_n(T) = \sum_i E_{n_i}(T_i) \geq \tfrac{1}{2}\sum_i \mathcal{E}_i n_i \geq \frac{\mathcal{E}}{2(1-\mathcal{E})}\big(n - E_n(T) - 1\big),$$

and solving gives

$$E_n(T) \geq \frac{\mathcal{E}}{2-\mathcal{E}}\,(n-1).$$

If $\mathcal{E}n > 2$, some algebra shows that the last quantity is greater than $\mathcal{E}n/2$, so the required inequality holds. Otherwise, $E_n(T) \geq 1 \geq \mathcal{E}n/2$. $\square$

Proof of Theorem 3. By Proposition 12, we may assume that r is b-free. By Theorem 1, it suffices to prove that $\liminf_{n\to\infty} E_n/n \geq \mathcal{E}$.

For any vertex $v \in V_o$, let $\mathcal{E}_v := \mathcal{E}(T_v)$, which is the probability that a simple random walk starting at v never visits v's parent. Also for any $v \in V$, let $h(v)$ be the probability that a random walk started at v ever hits o. Note that h is harmonic except at the root, where $h(o) = 1$, and that

$$h(v) = \prod_{x\in(o,v]} (1-\mathcal{E}_x).$$

where $(o,v]$ denotes the path from o to v (including v but not o). Also let

$$\ell_v := \liminf_{n\to\infty} \frac{E_n(T_v, r_v)}{n}.$$

and define by analogy with h the function

$$f(v) := \prod_{x\in(o,v]} (1-\ell_x).$$

We need to prove $\ell_\iota \geq \mathcal{E}_\iota$, which is equivalent to $f(\iota) \leq h(\iota)$. (In fact the argument will imply $f \equiv h$).

We first show that f is sub-harmonic except at o. By Lemma 25 we have for any $v \in V_o$,

$$\ell_v \geq 1 - \frac{1}{1+L}, \quad \text{where} \quad L = \sum_{i=1}^{b(v)} \ell_{v^{(i)}}.$$

(The $\mathcal{E}_v$'s satisfy the corresponding *equality*, which is equivalent to the harmonicity of h). Therefore, using the definition of f,

$$f(v^{(0)}) = \frac{f(v)}{1-\ell_v} \geq f(v)(1+L),$$

and so

$$\begin{aligned}\sum_{i=0}^{b(v)} f(v^{(i)}) &\geq f(v)(1+L) + \sum_{i=1}^{b(v)} f(v)(1-\ell_{v^{(i)}}) \\ &= f(v)(1+b(v)),\end{aligned}$$

which is the claimed sub-harmonicity.

Let ∂T be the set of ends of the tree T. Since h and f are decreasing along any end, we extend their definitions to ∂T via limits. We claim that

$$h(\eta) = 0 \implies f(\eta) = 0, \quad \eta \in \partial T. \tag{11}$$

To prove this, suppose $h(\eta) = 0$. Thus $\prod_{v\in\eta}(1-\mathcal{E}_v) = 0$, and so $\sum_{v\in\eta}\mathcal{E}_v = \infty$. Since by Lemma 26 we have $\ell_v \geq \mathcal{E}_v/2$ for all v, this implies $\sum_{v\in\eta}\ell_v = \infty$, and thus $f(\eta) = 0$.

Now let (X_t) be the simple symmetric walk on T, started at ι, and stopped at the first visit (if any) to o. Note that (X_t) almost surely either hits o or visits exactly one end η infinitely often – in that case we say the walk escapes to η. Let μ be the associated harmonic measure on $\{o\} \cup \partial T$, so $\mu(A)$ equals the probability that the walk escapes to some end in A. Now $(h(X_t))$ is a bounded martingale, therefore it converges, and $\mathbb{E}h(X_0) = \mathbb{E}\lim_{t\to\infty} h(X_t)$; that is,

$$\begin{aligned}h(\iota) &= \mu(\{o\}) \cdot h(o) + \int_{\partial T} h \, d\mu \\ &= h(\iota) \cdot 1 + \int_{\partial T} h \, d\mu.\end{aligned}$$

Thus, the last integral is 0, so $\mu\{\eta \in \partial T : h(\eta) > 0\} = 0$. (This fact also follows from [1, Proposition 8], for example). By (11), this implies $\mu\{\eta \in \partial T : f(\eta) > 0\} = 0$. On the other hand, $(f(X_t))$ is a bounded sub-martingale, so we obtain similarly

$$\begin{aligned}f(\iota) &\leq h(\iota) \cdot 1 + \int_{\partial T} f \, d\mu \\ &= h(\iota),\end{aligned}$$

completing the proof. ∎

11. Maximum depth

Proof of Theorem 7(i). By Theorem 6, if $\mathbb{E}R(v) \geq b-1$ then all particles return to the root a.s. Let V_n be the set of non-root vertices that have ever been visited when the nth particle returns to the root. Since a rotor always points to the last direction in which a particle left, at this time all vertices in V_n have their rotors pointing towards the root. It follows that particle $n+1$ will visit all vertices in V_n, as well as all their children. Let ΔV_n be the set of children of V_n that are not themselves elements of V_n, and note that $\#\Delta V_n = (b-1)\#V_n + 1$. Each time particle $n+1$ enters a new element of ΔV_n, it encounters a previously untouched subtree whose rotors are distributed as in the original tree. So before returning to V_n, it visits a number of new vertices that is equal in law to $\#V_1$. Thus

$$\#V_{n+1} = \#V_n + \sum_{i=1}^{(b-1)\#V_n+1} Z_i, \tag{12}$$

where (Z_i) are i.i.d. with the same law as $\#V_1$ and independent of V_n. (Thus, $(\#V_n)_{n\geq 1}$ is 'almost' a Galton-Watson process).

On the other hand, if a vertex v belongs to V_1, then so do all those of its children $v^{(k)}$ that satisfy $k > R(v)$ (and no others). Thus the graph induced by V_1 is a Galton-Watson tree with offspring distribution that of $b - R(v)$.

We write c_i, C_i for (small, large) constants in $(0,\infty)$ depending only on b and the distribution of $R(v)$. Since $\mathbb{E}R(v) = b-1$, the latter Galton-Watson tree is critical with bounded offspring distribution, therefore (see e.g. [16, §2.1: Theorem 2 and Lemma 4]) for all $N \geq 1$,

$$c_1/N \leq \mathbb{P}(\#V_1 > N^2) \leq C_1/N; \tag{13}$$

$$c_2/N \leq \mathbb{P}(D_1 > N) \leq C_2/N. \tag{14}$$

By (13), for (Z_i) i.i.d. with the same law as $\#V_1$,

$$\mathbb{P}\Big(\sum_1^N Z_i > N^2\Big) \geq \mathbb{P}\big(\max_1^N Z_i > N^2\big) \geq (1-c_1/N)^N \xrightarrow{N\to\infty} e^{-c_1},$$

therefore

$$\mathbb{P}\Big(\sum_1^N Z_i > N^2\Big) \geq c_3, \quad \forall N \geq 1.$$

(For large enough N this follows from the previous line, while the constant can be chosen so that it holds for small N because Z_1 has unbounded support). Since $(b-1)N+1 \geq N$, it follows from (12) that

$$\mathbb{P}\Big(\#V_{n+1} > (\#V_n)^2 \mid V_1, \ldots, V_n\Big) > c_3.$$

We have in any case $\#V_{n+1} > \#V_n$ and $\#V_2 \geq 2$, so by the law of large numbers,

$$\mathbb{P}\big(\#V_n \geq 2^{2^{(c_4 n)}}\big) \to 1 \quad \text{as } n \to \infty. \tag{15}$$

We now use a similar argument to convert (15) to a lower bound on the depth. Recall that D_n is the maximum level of all vertices in V_n, and note that

$$D_{n+1} \geq \max_{i=1}^{(b-1)\#V_n+1} Y_i,$$

where (Y_i) are the depths of the subtrees added to V_n to form V_{n+1}, which are i.i.d. with the same law as D_1. By (14), $\mathbb{P}(\max_i^N Y_i > \sqrt{N}) \to 1$ as $N \to \infty$, so $\mathbb{P}\big(D_{n+1} \geq \sqrt{\#V_n}\big) \to 1$ as $n \to \infty$, which together with (15) gives $\mathbb{P}\big(D_n \geq e^{e^{cn}}\big) \to 1$.

Turning to the upper bound, by (13) we have for all $t > 0$ and $N \geq 1$,

$$\mathbb{P}\bigg(\frac{\log \sum_1^N Z_i}{\log N} > t\bigg) = \mathbb{P}\bigg(\sum_1^N Z_i > N^t\bigg) \leq N\mathbb{P}(Z_1 > N^{t-1}) \leq C_1 N^{-\frac{t-3}{2}}.$$

Hence (by the above for $N \geq 2$ and (13) for $N = 1$),

$$\mathbb{E}\bigg(\frac{\log \sum_1^N Z_i}{\log N}\bigg) \leq C_3.$$

Thus from (12), $\mathbb{E} \log \#V_n \leq (C_4)^n$, so Markov's inequality gives $\mathbb{P}\big(\#V_n > e^{e^{Cn}}\big) \to 0$ as $n \to \infty$. Finally, D_n satisfies the same bound, since $D_n \leq \#V_n$. ☐

Proof of Theorem 7(ii). The lower bound $n \leq D_n$ clearly holds regardless of the rotor configuration, so we turn to the upper bound. As in part (i), a.s. no particle escapes to infinity, and each particle visits all vertices visited by its predecessor.

Let x be a vertex of $\mathbb{T}_b$ at level $h+1$, and watch the rotor walk only while it is on the path $\pi = \pi(x)$ from o to x. Since whenever the particle leaves π it always returns via the same vertex, its behaviour on π is identical to that of a rotor walk on a path, as determined by the initial rotor positions on π. If π has vertices $o = x_0, x_1, \ldots, x_h, x_{h+1} = x$, let $k_i = k_i(x)$ be such that $x_{i+1} = (x_i)^{(k_i)}$ for $i = 1, \ldots, h$. Then the nth particle reaches x if and only if

$$\sum_{i=1}^{h} \mathbb{1}[R(x_i) \geq k_i(x)] < n.$$

Now let X be a uniformly randomly chosen vertex at level $h+1$, independent of the initial rotor configuration. Then $(k_i(X))_{i=1}^h$ are i.i.d. uniformly random on $\{1,\ldots,b\}$. If K is uniform on $\{1,\ldots,b\}$ and independent of $R(v)$ then $\mathbb{P}(R(v) \geq K) = \mathbb{E}R(v)/b =: p$, say. By the above, it follows that the probability particle n reaches X is the probability that a Binomial(h,p) random variable B is less than n.

Let Z be the number of vertices at level $h+1$ visited by particle n. Then

$$\mathbb{P}(D_n > h) = \mathbb{P}(Z > 0) \leq \mathbb{E}Z = b^h\, \mathbb{P}(B < n). \tag{16}$$

On the other hand, for $\alpha \in (0,1)$, by a standard Chernoff bound (i.e. $\mathbb{P}(B < \alpha h) \leq s^{-\alpha h}\mathbb{E}s^B$ with $s = (p^{-1}-1)/(\alpha^{-1}-1)$),

$$\mathbb{P}(B < \alpha h) \leq \left[\left(\frac{p}{\alpha}\right)^{\alpha}\left(\frac{1-p}{1-\alpha}\right)^{1-\alpha}\right]^h.$$

We have

$$b\left(\frac{p}{\alpha}\right)^{\alpha}\left(\frac{1-p}{1-\alpha}\right)^{1-\alpha} \xrightarrow{\alpha\to 0} b(1-p) = b - \mathbb{E}R(v) < 1.$$

Therefore we may choose $\alpha > 0$ (depending only on b and $\mathbb{E}R(v)$) so that the left side is less than 1. Then from (16), $\mathbb{P}(D_{\lfloor \alpha h\rfloor} > h) \to 0$ as $h \to \infty$, hence $\mathbb{P}(D_n > Cn) \to 0$ as $n \to \infty$, as required. $\square$

Open Problems

(i) Characterize the set of possible escape sequences $e(T,\mathbf{0})$, where T varies over all trees (with root of degree 1 and all degrees finite), and $\mathbf{0}$ is the initial configuration in which all rotors point towards the root.

(ii) If $e = e(T,r)$ is an escape sequence for a tree T (with some initial rotor configuration r), must every sequence e' satisfying $e' \leq e$ also be an escape sequence for T?

References

[1] I. Benjamini and Y. Peres. Random walks on a tree and capacity in the interval. *Ann. Inst. H. Poincaré Probab. Statist.*, 28(4):557–592, 1992.

[2] J. Cooper, B. Doerr, T. Friedrich, and J. Spencer. Deterministic random walks on regular trees. In *Proceedings of SODA*, pages 766–772, 2008.

[3] J. Cooper, B. Doerr, J. Spencer, and G. Tardos. Deterministic random walks. In *Proceedings of the Workshop on Analytic Algorithmics and Combinatorics*, pages 185–197, 2006.

[4] J. Cooper, B. Doerr, J. Spencer, and G. Tardos. Deterministic random walks on the integers. *European J. Combin.*, 28(8):2072–2090, 2007.

[5] J. N. Cooper and J. Spencer. Simulating a random walk with constant error. *Combin. Probab. Comput.*, 15(6):815–822, 2006.

[6] B. Doerr and T. Friedrich. Deterministic random walks on the two-dimensional grid. In *Combinatorics, Probability and Computing*, volume 18, pages 123–144. Cambridge University Press, 2009.

[7] B. Doerr, T. Friedrich, and T. Sauerwald. Quasirandom rumor spreading. In *Proceedings of SODA*, pages 773–781, New York, 2008. ACM.

[8] I. Dumitriu, P. Tetali, and P. Winkler. On playing golf with two balls. *SIAM J. Discrete Math.*, 16(4):604–615 (electronic), 2003.

[9] R. Durrett. *Probability: theory and examples.* Duxbury Press, Belmont, CA, second edition, 1996.

[10] T. Friedrich, M. Gairing, and T. Sauerwald. Quasirandom rumor spreading. In *Proceedings of SODA*, pages 1620–1629, New York, 2010. ACM.

[11] T. Friedrich and L. Levine. Fast simulation of large-scale growth models, 2010, arXiv:1006.1003.

[12] A. E. Holroyd, L. Levine, K. Mészáros, Y. Peres, J. Propp, and D. B. Wilson. Chip-firing and rotor-routing on directed graphs. In V. Sidoravicius and M. E. Vares, editors, *In and Out of Equilibrium 2*, volume 60 of *Progress in Probability*, pages 331–364. Birkhäuser, 2008.

[13] A. E. Holroyd and J. Propp. Rotor walks and Markov chains. In M. Lladser, R. S. Maier, M. Mishna, and A. Rechnitzer, editors, *Algorithmic Probability and Combinatorics*, volume 520 of *Contemporary Mathematics*, pages 105–125. Amer. Math. Soc., 2010.

[14] W. Kager and L. Levine. Rotor-router aggregation on the layered square lattice, 2010, arXiv:1003.4017.

[15] M. Kleber. Goldbug variations. *Math. Intelligencer*, 27(1):55–63, 2005.

[16] V. F. Kolchin. *Random mappings.* Translation Series in Mathematics and Engineering. Optimization Software Inc. Publications Division, New York, 1986. Translated from the Russian, With a foreword by S. R. S. Varadhan.

[17] I. Landau and L. Levine. The rotor-router model on regular trees. *J. Combin. Theory Ser. A*, 116(2):421–433, 2009.

[18] L. Levine. The sandpile group of a tree. *European J. Combin.*, 30(4):1026–1035, 2009.

[19] L. Levine and Y. Peres. Spherical asymptotics for the rotor-router model in Z^d. *Indiana Univ. Math. J.*, 57(1):431–449, 2008.

[20] L. Levine and Y. Peres. Strong spherical asymptotics for rotor-router aggregation and the divisible sandpile. *Potential Analysis*, 30(1):1–27, 2009, arXiv:0704.0688.

[21] L. Levine and Y. Peres. Scaling limits for internal aggregation models with multiple sources. *J. d'Analyse Math.*, to appear, 2010, arXiv:0712.3378.

[22] V. B. Priezzhev, D. Dhar, A. Dhar, and S. Krishnamurthy. Eulerian walkers as a model of self-organized criticality. *Phys. Rev. Lett.*, 77:5079–5082, 1996.

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