# Discrete Low-Discrepancy Sequences

Omer Angel Alexander E. Holroyd
James B. Martin James Propp

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**Abstract**

Holroyd and Propp used Hall's marriage theorem to show that, given a probability distribution $\pi$ on a finite set $\mathcal{S}$, there exists an infinite sequence $s_1, s_2, \ldots$ in $\mathcal{S}$ such that for all integers $k \geq 1$ and all $s$ in $\mathcal{S}$, the number of $i$ in $[1, k]$ with $s_i = s$ differs from $k\,\pi(s)$ by at most 1. We prove a generalization of this result using a simple explicit algorithm. A special case of this algorithm yields an extension of Holroyd and Propp's result to the case of discrete probability distributions on infinite sets.

Recently there has been an upsurge of interest in non-random processes that mimic interesting aspects of random processes, where the fidelity of the mimicry is a consequence of discrepancy constraints built into the constructions (for a general survey of discrepancy theory, see [1]). A recent example is the work of Friedrich, Gairing and Sauerwald [7] on load-balancing; other examples, linked by their use of the "rotor-router mechanism", are the work of Cooper, Doerr, Friedrich, Spencer, and Tardos [2, 3, 4, 5, 6] on derandomized random walk on grids ("$P$-machines"), the work of Landau, Levine and Peres [9, 11, 12, 13] on derandomized internal diffusion-limited aggregation on grids and trees, and the work of Holroyd and Propp [8] on derandomized Markov chains. Here we focus on derandomizing something even more fundamental to probability theory: the notion of an independent sequence of discrete random variables.

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Given a discrete probability distribution $\pi(\cdot)$ on a set $\mathcal{S}$, the associated i.i.d. process satisfies the law of large numbers. That is, if we choose $S_1, S_2, \dots$ from $\mathcal{S}$ independently at random in accordance with $\pi$, the random variables $N_k(s) := \#\{i : 1 \le i \le k \text{ and } S_i = s\}$ have the property that $N_k(s)/k \to \pi(s)$ almost surely as $k \to \infty$, and indeed the discrepancies $N_k(s) - k\,\pi(s)$ are typically $O(\sqrt{k})$. A derandomized analogue of an i.i.d. process should have the property that $N_k(s) - k\,\pi(s)$ is $o(k)$, and derandomized processes with $|N_k(s) - k\,\pi(s)|$ as small as possible are especially interesting. It is too much to ask that the unscaled differences $N_k(s) - k\,\pi(s)$ themselves go to zero (since $N_k(s)$ is always an integer while $k\,\pi(s)$ typically is not), but we can ask that these differences stay bounded.

Holroyd and Propp [8] used such derandomized i.i.d. processes ("low-discrepancy stacks", in their terminology) in order to make their theory applicable to Markov chains with irrational transition probabilities. Indeed, the following theorem appears (with slightly different notation) as Proposition 11 in [8].

**Theorem 1** (Low-discrepancy sequences for i.i.d. processes; [8])**.** *Given a probability distribution $\pi$ on a finite set $\mathcal{S}$, there exists an infinite sequence $s_1, s_2, \dots$ in $\mathcal{S}$ such that for all $k \ge 1$ and all $s$ in $\mathcal{S}$, the number of $i$ in $[1, k]$ with $s_i = s$ differs from $k\,\pi(s)$ by at most 1.*

Here we give a proof of this result that is simultaneously simpler and more constructive than Holroyd and Propp's, and applies even when the set $\mathcal{S}$ is infinite. Furthermore, our construction gives a simple way to derandomize sequences of discrete random variables that are independent but not identically distributed.

**Theorem 2** (Low-discrepancy sequences for independent processes)**.** *Given discrete probability distributions $\pi_1, \pi_2, \dots$ on some countable set $\mathcal{S}$, there exists an infinite sequence $s_1, s_2, \dots$ in $\mathcal{S}$ such that for all $k \ge 1$ and all $s$ in $\mathcal{S}$, the quantities $N_k(s) := \#\{i : 1 \le i \le k \text{ and } s_i = s\}$ and $P_k(s) := \sum_{i=1}^{k} \pi_i(s)$ differ in absolute value by strictly less than 1.*

Theorem 1 is the special case of Theorem 2 in which $\mathcal{S}$ is finite and $\pi_1 = \pi_2 = \cdots = \pi$, with an infinitesimally weaker inequality in the conclusion.

**Remark.** For every choice of $s_1, s_2, \dots$, we have $\sum_s N_k(s) = k = \sum_s P_k(s)$, whence $\sum_s (N_k(s) - P_k(s)) = 0$. Moreover, if we were to choose $S_1, S_2, \dots$

independently from $\mathcal{S}$ in accordance with the respective probability distributions $\pi_i$, then the expected number of $i$ in $[1, k]$ with $S_i = s$ would be $\sum_{i=1}^{k} \pi_i(s)$, so the expected value of $N_k(s) - P_k(s)$ would be zero for each $s$.

*Proof of Theorem 2.* We present an algorithm for determining the sequence $(s_k)$. Our algorithm is as follows. Given $s_1, \ldots, s_k$ (with $k \geq 0$), let $s_{k+1}$ be the candidate $s$ with the earliest deadline, where we say $s$ is a **candidate** (for being the $(k+1)$st term) if $N_k(s) - P_{k+1}(s) < 0$, and where we define the **deadline** for such an $s$ as the smallest integer $k' \geq k+1$ for which $N_k(s) - P_{k'}(s) \leq -1$. Ties may be resolved in any fashion.

For $k \geq 0$ write $D_k(s) := N_k(s) - P_k(s)$ (note that $D_0(s) = 0$). First observe that $s$ is a candidate if and only if taking $s_{k+1} = s$ would lead to $D_{k+1}(s) < 1$; equivalently, $s$ fails to be a candidate if and only if taking $s_{k+1} = s$ would lead to $D_{k+1}(s) \geq 1$. That is, when we are choosing the $(k+1)$st term of the sequence, $s$ fails to be a candidate if and only if choosing $s_{k+1}$ to be $s$ would cause $s$ to be **oversampled** from time 1 to time $k+1$. It is clear that there is always at least one candidate, since $\sum_s (N_k(s) - P_{k+1}(s)) = -1 < 0$.

Also note that if $s$ is a candidate with deadline $k'$, then taking $s_{k+1}, \ldots, s_{k'}$ all unequal to $s$ would lead to $D_{k'}(s) \leq -1$; that is, such an $s$ would be **undersampled** from time 1 to time $k'$ if it were not chosen to be at least one of $s_{k+1}, \ldots, s_{k'}$.

For $k' > k \geq 0$ define $R_{k,k'}(s) := \lfloor P_{k'}(s) - N_k(s) \rfloor^+$ (where $x^+ := \max(x, 0)$). Thus $R_{k,k'}(s)$ is the minimal number of the terms $s_{k+1}, \ldots, s_{k'}$ that must be equal to $s$ in order to prevent $s$ from being undersampled from time 1 to time $k'$. If $R_{k,k+1}(s) = 1$, then $s_{k+1}$ must be chosen to equal $s$ in order to prevent $s$ from being undersampled from time 1 to time $k+1$; we call such an $s$ **critical**. We will show by induction on $k \geq 0$ that $\sum_s R_{k,k'}(s) \leq k' - k$ for all $k' > k$. This implies in particular that $\sum_s R_{k,k+1}(s) \leq 1$ for all $k$, so that at each step at most one $s$ is critical; this in turn implies that no $s$ is ever undersampled. And since our procedure only chooses candidates, no $s$ is ever oversampled either.

First, consider $k = 0$: we have $D_0(s) = 0$ and $R_{0,k'}(s) = \lfloor P_{k'}(s) \rfloor$, so $\sum_s R_{0,k'}(s) = \sum_s \lfloor P_{k'}(s) \rfloor \leq \lfloor \sum_s P_{k'}(s) \rfloor = k' - 0$ as claimed.

Now take $k \geq 0$, and suppose for induction that $\sum_s R_{k,k'}(s) \leq k' - k$ for some particular $k' > k+1$. We wish to show that $\sum_s R_{k+1,k'}(s) \leq k' - (k+1)$. There are two cases to consider. First, if $R_{k,k'}(s_{k+1}) > 0$, then $R_{k+1,k'}(s_{k+1}) = R_{k,k'}(s_{k+1}) - 1$ and $R_{k+1,k'}(s) = R_{k,k'}(s)$ for all $s \neq s_{k+1}$, so $\sum_s R_{k+1,k'}(s) = (\sum_s R_{k,k'}(s)) - 1 \leq (k' - k) - 1 = k' - (k+1)$ as claimed.

Second, if $R_{k,k'}(s_{k+1}) = 0$, then the deadline for $s_{k+1}$ is greater than $k'$. Since our algorithm chooses $s_{k+1}$ as the $(k+1)$st term, $s_{k+1}$ must be the candidate with the earliest deadline. This means that no $s \neq s_{k+1}$ in $\mathcal{S}$ with $R_{k,k'}(s) > 0$ is a candidate; that is, every $s \neq s_{k+1}$ with $R_{k,k'}(s) > 0$ must satisfy $N_k(s) \geq P_{k+1}(s)$, and since $N_{k+1}(s) = N_k(s)$, we must have $D_{k+1}(s) \geq 0$, implying $R_{k+1,k'}(s) = \lfloor -D_{k+1}(s) + \sum_{i=k+2}^{k'} \pi_i(s) \rfloor^+ \leq \lfloor \sum_{i=k+2}^{k'} \pi_i(s) \rfloor^+ \leq \sum_{i=k+2}^{k'} \pi_i(s)$. Likewise, for all $s$ with $R_{k,k'}(s) = 0$, we have $R_{k+1,k'}(s) = 0$, implying $R_{k+1,k'}(s) \leq \sum_{i=k+2}^{k'} \pi_i(s)$. Hence $\sum_s R_{k+1,k'}(s) \leq \sum_s \sum_{i=k+2}^{k'} \pi_i(s) = \sum_{i=k+2}^{k'} \sum_s \pi_i(s) = \sum_{i=k+2}^{k'} 1 = k' - (k+1)$, as claimed. $\square$

In reading the proof, the reader may find it helpful to imagine the following scenario. Let $\mathcal{S}$ be a set of creatures, each of which has a **surplus** (or "energy level") that is initially 0. At each step a single creature gets fed. At the $k$th step, the surplus $D_k(s)$ of creature $s$ decreases by $\pi_k(s)$, but in addition increases by 1 (giving a net change of $1 - \pi_k(s)$) if $s$ gets fed. After each step the sum of the surpluses is zero. If a creature's surplus ever falls to $-1$ or less, the creature dies of starvation; if its surplus ever rises to 1 or more, it dies of overfeeding. Our strategy for keeping all the creatures alive is to always feed the creature that if left unfed would die earliest of starvation, excepting those that cannot be fed because they would immediately die of overfeeding.

Independently of our work, in the context of a one-sided version of the discrepancy-control problem arising from an email post by John Lee [10], Oded Schramm and Fedja Nazarov considered other algorithms for keeping the quantities $k\,\pi(s) - N_k(s)$ from becoming too large in the case where all the $\pi_i$ equal $\pi$.

Considering the sequence of discrepancy vectors $(D_k)$ leads to the following reformulation of Theorem 1.

**Corollary 3.** *For any probability vector $\pi \in \mathbb{R}^n$ there is a compact $K \subseteq [-1,1]^n$ containing $(0,0,\ldots,0)$ such that $K \subseteq \bigcup_{i=1}^n (K + \pi - e_i)$, where $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathbb{R}^n$.*

To see that this implies Theorem 1, note that, given $K$ and $D_k$ (with $k \geq 0$), we can choose $s_{k+1} = i$ such that $D_k - \pi + e_i \in K$; this choice guarantees that the discrepancy vector $D_{k+1}$ lies in $K$ and hence in $[-1,1]^n$. Conversely, note that if we take the sequence $s_1, s_2, \ldots$ given by Theorem 1, the bounded set $\{D_k : \ k \geq 0\}$ satisfies all the conditions of Corollary 3 except for compactness. Hence we can prove Corollary 3 by taking $K$ to be the closure of $\{D_k : \ k \geq 0\}$. Corollary 3 asserts the existence of a set $K$ containing the origin that is covered by translations of itself by given vectors $v_i := \pi - e_i$. Since the zero vector is in the convex hull of the $v_i$, a sufficiently large ball in the subspace spanned by the $v_i$ achieves this, but Corollary 3 provides the bound $K \subseteq [-1, 1]^n$.

The constant 1 in Theorem 2 cannot be improved; that is, there is no $c < 1$ with the property that for all $\pi_1, \pi_2, \ldots$ there exists a way to choose $S_1, S_2, \ldots$ from $\mathcal{S}$ so that the discrepancies $N_k(s)/k - \pi(s)$ all stay within the interval $(-c, +c)$. Consider for instance the case where each $\pi_i$ is the uniform distribution on a finite set $\mathcal{S}$ of cardinality $n$, and take $k = n - 1$. There exists $s$ in $\mathcal{S}$ distinct from $s_1, \ldots, s_k$ and this $s$ satisfies $P_k(s) - N_k(s) = k/n - 0 = 1 - 1/n$. Although this example shows that the constant 1 cannot be improved, it is possible that there is a universal strict subset $K$ of $(-1, +1) \times (-1, +1) \times \cdots$ with the property that for all $\pi_1, \pi_2, \ldots$ there exists a way of choosing $S_1, S_2, \ldots$ from $\mathcal{S}$ so that the vector of discrepancies stays within the set $K$.

It is also possible that Theorem 2 might be strengthened by controlling the discrepancies between $N_k(s) - N_j(s)$ and $\pi_{j+1}(s) + \pi_{j+2}(s) + \cdots + \pi_k(s)$ for all $j, k$ with $1 \leq j \leq k$ and all $s$ in $\mathcal{S}$. It is easy to deduce from Theorem 2 that every such discrepancy has absolute value less than 2 (since it is just $D_k(s) - D_j(s)$), but perhaps one can show that there exists a way of choosing $S_1, S_2, \ldots$ so that every such discrepancy has absolute value less than 1.

In determining $s_k$, our algorithm typically requires knowledge of the future distributions $(\pi_i)_{i \geq k}$ (actually a finite but unbounded number of them). This is unavoidable, as may be seen by the following example. Let $\pi_k$ be uniform on $1, \ldots, 5$ for each of $k = 1, 2, 3$. Regardless of $s_1, s_2, s_3$ there will be two $i$'s with $N_3(i) - P_3(i) = -0.6$. If $s_1, s_2, s_3$ are chosen in ignorance of $\pi_4$, it is possible for $\pi_4$ to be uniform on those two $i$'s. Then any choice of $s_4$ will result in $N_4(i) - P_4(i) = -1.1$ for some $i$. With more than 5 values the discrepancies can be even larger in magnitude. It might be interesting to know how good a bound on discrepancy can be achieved by algorithms that are constrained to have $s_n$ depend only on $\pi_1, \ldots, \pi_n$. Of course, when the $\pi_i$'s are all equal as in Theorem 1 this issue is nonexistent.

One way in which Theorem 1 might be strengthened is by finding a construction that minimizes $\max_k \sum_{i<k} f(s_i)$, where $f$ is some function on $\mathcal{S}$ satisfying $\sum_s f(s)\pi(s) = 0$. Sums of the form $\max_k \sum_{i<k} f(s_i)$ play an important role in [8]; there the elements $s$ of $\mathcal{S}$ correspond to transitions $u \to v$ in a

Markov chain (with $u$ fixed and $v$ varying), $\pi(s)$ equals the transition probability $p(u,v)$, and $f(s)$ equals $h(v)-h(u)$ where the function $h(\cdot)$ is harmonic at $u$ (i.e., $\sum_v p(u,v)h(v) = h(u)$), implying $\sum_s f(s)\pi(s) = 0$. For example, consider random walk on $\mathbb{Z}^2$, where a vertex $u$ has four neighbors $u_N$, $u_S$, $u_E$, and $u_W$ with $p(u_N) = p(u_S) = p(u_E) = p(u_W) = \frac{1}{4}$. Key results in [8] treat rotor-routers that rotate in the repeating pattern $N, E, S, W, N, E, S, W, \ldots$ and show that the resulting rotor-router walks closely mimic certain features of the random walk. However, since the relevant discrepancies are controlled by quantities of the form $\max_k \sum_{i<k} f(s_i)$, and since the function $h$ has the property that $h(u_N) + h(u_S)$ and $h(u_E) + h(u_W)$ are close to $2h(u)$ (implying that $f(N) + f(S)$ and $f(E) + f(W)$ are close to zero), there is reason to think that rotor-routers that rotate in the repeating pattern $N, S, E, W, N, S, E, W, \ldots$ would give smaller discrepancy for the quantities of interest.

As an important special case of Theorem 1, suppose $\pi$ is rational. It is then natural to ask that the sequence $s_1, s_2, \ldots$ be periodic (so that one has a "rotor" in the sense of [8]). Our algorithm as described does not guarantee periodicity, because we allowed ties to be broken arbitrarily. However, if we add the stipulation that ties are always broken in some pre-determined way depending only on the deadlines and the discrepancies $D_k(s)$, then our algorithm yields a periodic sequence, with period equal to the least common multiple $m$ of the denominators of the rational numbers $\pi(s)$. Indeed, it is clear that the sequence generated by the algorithm is eventually periodic, and that the period cannot be less than $m$. On the other hand, the construction gives $|N_m(s) - P_m(s)| < 1$; but $N_m(s)$ and $P_m(s)$ are both integers, so they must be equal. Hence $D_m(s) = 0 = D_0(s)$ for all $s$, so the procedure enters a loop at time $m$.

Theorem 1 can be rephrased as follows: for any sequence of non-negative real numbers $\pi(1), \pi(2), \ldots$ summing to 1, there exists a partition of the natural numbers into sets $F_1, F_2, \ldots$ where for all $i$ the set $F_i$ has density $\pi(i)$, and $|F_i \cap \{1, \ldots, k\}| - \pi(i)k$ lies in $[-1, 1]$ for all $k \geq 1$ (of course the second property implies the first). An even stronger condition we might seek is that the gap between the $m$th and $n$th elements of $F_i$ (for all $i$ and all $n \geq m \geq 1$) is within 1 of $(n-m)/\pi(i)$. If "within 1" is interpreted in the strict sense (i.e., the difference is strictly less than 1), then this condition cannot always be achieved; e.g., with $\pi = (\frac{1}{2}, \frac{1}{3}, \frac{1}{6})$, the only way to satisfy the condition would be to partition the natural numbers into three arithmetic progressions with densities $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{6}$, which clearly cannot be done. However, if "within

1” is interpreted in the weak sense (i.e., the difference is less than or equal to 1), then we do not know of a counterexample.

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## References

[1] B. Chazelle. *The Discrepancy Method: Randomness and Complexity*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, 2000.

[2] J. Cooper, B. Doerr, T. Friedrich, and J. Spencer. Deterministic random walks on regular trees. In *Proceedings of SODA 2008*, pages 766–772, 2008.

[3] J. Cooper, B. Doerr, J. Spencer, and G. Tardos. Deterministic random walks. In *Proceedings of the Workshop on Analytic Algorithmics and Combinatorics*, pages 185–197, 2006.

[4] J. Cooper, B. Doerr, J. Spencer, and G. Tardos. Deterministic random walks on the integers. *European J. Combin.*, 28(8):2072–2090, 2007.

[5] J. N. Cooper and J. Spencer. Simulating a random walk with constant error. *Combin. Probab. Comput.*, 15(6):815–822, 2006.

[6] B. Doerr and T. Friedrich. Deterministic random walks on the two-dimensional grid. In *Combinatorics, Probability and Computing*, volume 18, pages 123–144. Cambridge University Press, 2009.

[7] T. Friedrich, M. Gairing, and T. Sauerwald. Quasirandom load balancing. To appear in *Proceedings of SODA 2010*.

[8] A. E. Holroyd and J. Propp. Rotor walks and Markov chains. To appear in *Algorithmic Probability and Combinatorics*, arXiv:0904.4507.

[9] I. Landau and L. Levine. The rotor-router model on regular trees. *J. Combin. Theory Ser. A*, 116:421–433, 2009.

[10] J. Lee. Lower and upper bound of variables (1D and 2D cases). 2001. `http://www.cut-the-knot.org/exchange/multitasking.shtml`.

[11] L. Levine and Y. Peres. Scaling limits for internal aggregation models with multiple sources. *J. d'Analyse Math.*, to appear, 2007, arXiv:0712.3378.

[12] L. Levine and Y. Peres. Spherical asymptotics for the rotor-router model in $Z^d$. *Indiana Univ. Math. J.*, 57(1):431–449, 2008.

[13] L. Levine and Y. Peres. Strong spherical asymptotics for rotor-router aggregation and the divisible sandpile. *Potential Analysis*, 30(1):1–27, 2009, arXiv:0704.0688.

Omer Angel: `angel at math dot ubc dot ca`
Dept. of Mathematics, University of British Columbia, Vancouver, BC, Canada.

Alexander E. Holroyd: `holroyd at math dot ubc dot ca`
Microsoft Research, 1 Microsoft Way, Redmond WA, USA; and
University of British Columbia, 121-1984 Mathematics Road, Vancouver, BC, Canada.

James B. Martin: `martin at stats dot ox dot ac dot uk`
Department of Statistics, University of Oxford, 1 South Parks Road, Oxford, UK.

James Propp: `jpropp at cs dot uml dot edu`
University of Massachussets Lowell, 1 University Ave., Olney Hall 428, Lowell, MA, USA.