SYMMETRIC 1−DEPENDENT COLORINGS OF THE INTEGERS

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Abstract. In a recent paper by the same authors, we constructed a stationary 1−dependent 4−coloring of the integers that is invariant under permutations of the colors. This was the first stationary k−dependent q−coloring for any k and q. When the analogous construction is carried out for $q > 4$ colors, the resulting process is not k−dependent for any k. We construct here a process that is symmetric in the colors and 1−dependent for every $q \geq 4$. The construction uses a recursion involving Chebyshev polynomials evaluated at $\sqrt{q}/2$.

1. Introduction

By a (proper) q−coloring of the integers, we mean a sequence $(X_i : i \in \mathbb{Z})$ of $[q]$−valued random variables satisfying $X_i \neq X_{i+1}$ for all i (where $[q] := \{1, \ldots, q\}$). The coloring is said to be stationary if the (joint) distribution of $(X_i : i \in \mathbb{Z})$ agrees with that of $(X_{i+1} : i \in \mathbb{Z})$, and k−dependent if the families $(X_i : i \leq m)$ and $(X_i : i > m + k)$ are independent of each other for each m. In [2], we gave a construction of a stationary 1−dependent 4−coloring of the integers that is invariant under permutations of the colors. When the same construction is carried out for $q > 4$ colors, the resulting distribution is not k−dependent for any k. Of course, the 1−dependent 4−coloring is also a 1−dependent q−coloring for every $q > 4$, and one may obtain other 1−dependent q−colorings by splitting a color into further colors using an independent source of randomness. However, these colorings are not symmetric in the colors. We give here a modification of the process of [2] that is symmetric in the colors and 1−dependent for every $q \geq 4$. Here is our main result.

Theorem 1. *For each integer $q \geq 4$, there exists a stationary 1−dependent q−coloring of the integers that is invariant in law under permutations of the colors and under the reflection $(X_i : i \in \mathbb{Z}) \mapsto (X_{-i} : i \in \mathbb{Z})$.*

Our construction is given in the next section. Sections 3 and 4 provide some preliminary results and the proof of Theorem 1 respectively.

2. The construction

For $x = (x_1, x_2, \ldots, x_n) \in [q]^n$, we will write $P(x) = \mathbb{P}(X_1 = x_1, \ldots, X_n = x_n)$. To motivate the construction, we begin by noting that the finite-dimensional distributions P of the 4−coloring in [2] are defined recursively by $P(\emptyset) = 1$ and

$$P(x) = \frac{1}{2(n+1)} \sum_{i=1}^{n} P(\widehat{x}_i) \tag{1}$$

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for proper $x \in [4]^n$, where $\widehat{x}_i$ is obtained from x by deleting the ith entry in x. Of course, even if x is proper, $\widehat{x}_i$ may not be. So the definition is completed by setting $P(x) = 0$ for x's that are not proper.

For general $q \geq 4$, we will now allow the coefficients in the defining sum to depend on i as well as n. Considering many special cases, and the constraints imposed by the 1−dependence requirement, we were led to define

$$P(x) = \frac{1}{D(n+1)} \sum_{i=1}^{n} C(n-2i+1) P(\widehat{x}_i) \tag{2}$$

for proper $x \in [4]^n$, in terms of two sequences C and D. Again motivated by computations in special cases, we take

$$\begin{aligned} C(n) &= T_n\big(\sqrt{q}/2\big), && n \geq 0; \\ D(n) &= \sqrt{q}\, U_{n-1}\big(\sqrt{q}/2\big), && n \geq 1, \end{aligned}$$

where T_n and U_n are the Chebyshev polynomials of the first and second kind respectively.

There are several standard equivalent definitions of Chebyshev polynomials. One is

$$T_n(u) = \cosh(nt) \quad \text{and} \quad U_n(u) = \frac{\sinh[(n+1)t]}{\sinh(t)}, \quad \text{where } u = \cosh(t). \tag{3}$$

A variant definition using trigonometric functions (e.g. (22:3:3-4) of [3]) is easily seen to be equivalent by taking t imaginary; the hyperbolic function version is convenient for arguments $u \geq 1$. Another definition is

$$T_n(u) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} u^{n-2k}(u^2-1)^k \quad \text{and} \quad U_n(u) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+1}{2k+1} u^{n-2k}(u^2-1)^k.$$

That this is equivalent to (3) follows from e.g. (22:3:1-2) of [3].

If x is not a proper coloring, we take $P(x) = 0$ as before. We extend both sequences C and D to all integer arguments by taking $C(n)$ and $D(n)$ to be even and odd functions of n respectively (in accordance with (3)).

Observe that $C(n)$ and $D(n)$ are strictly positive for $q \geq 4$ and $n \geq 1$, and therefore $P(x)$ is strictly positive for all proper x. Note also that $C(n-2i+1)/D(n+1)$ is rational; therefore so is $P(x)$. (The factors of $\sqrt{q}$ cancel). When $q = 4$ we have $C(n) = 1$ and $D(n) = 2n$, and so (2) reduces to (1) in this case. As we will see, the fact that the coefficients in (2) depend on i substantially complicates the verifications of the required properties of P.

Here are a few examples of cylinder probabilities generated by (2).

$$P(1) = \frac{1}{q}, \quad P(12) = \frac{1}{q(q-1)}, \quad P(121) = \frac{1}{q^2(q-1)}, \quad P(123) = \frac{1}{q^2(q-2)},$$
$$P(1212) = \frac{q-3}{q^2(q-1)(q^2-3q+1)}, \quad P(1234) = \frac{1}{q^2(q^2-3q+1)}.$$

3. Preliminary results

Chebyshev polynomials satisfy a number of standard identities. They lead to identities satisfied by the sequences C and D. The first three in the proposition below are examples of this. The fourth is a consequence of the third one. Before stating them, we record some values of C and D to facilitate checking computations here and later.

$$C(0) = 1, \quad C(1) = \frac{\sqrt{q}}{2}, \quad C(2) = \frac{q-2}{2}, \quad C(3) = \frac{\sqrt{q}(q-3)}{2}, \quad C(4) = \frac{q^2-4q+2}{2}.$$
$$D(0) = 0, \quad D(1) = \sqrt{q}, \quad D(2) = q, \quad D(3) = \sqrt{q}(q-1), \quad D(4) = q(q-2).$$

Proposition 2. *For $j, k, \ell, m, n \in \mathbb{Z}$, the following identities hold.*

$$2C(m)C(n) = C(m+n) + C(n-m). \tag{4}$$

$$\frac{q-4}{2q} D(m)D(n) = C(m+n) - C(n-m). \tag{5}$$

$$2C(m)D(n) = D(m+n) + D(n-m). \tag{6}$$

$$C(j+k)D(k+\ell) = C(k)D(j+k+\ell) - C(\ell)D(j). \tag{7}$$

Proof. The first three parts are immediate consequences of (22:5:5-7) in [3], or 22.7.24-26 in [1], if m and n are nonnegative. None of the identities is changed by changing the sign of either m or n. Therefore, they hold for all m and n. Alternatively, the identities may be checked directly from (3) using the product formulae for hyperbolic functions. For (7), replace the products of C's and D's by sums of D's using (6), and then use the fact that D is an odd function. □

Next we verify some identities that involve both the sequences C and D and the measure P defined by (2). For the statement of the second part of the next result, let

$$Q(x) = \frac{1}{D(n+1)} \sum_{i=1}^{n} C(2i) P(\widehat{x}_i) \quad \text{and} \quad Q^*(x) = \frac{1}{D(n+1)} \sum_{i=1}^{n} C(2n-2i+2) P(\widehat{x}_i)$$

for $x \in [q]^n$. The first part of Proposition 3 is needed in proving the second part, which plays a key role in the proof of consistency and 1−dependence of P. Note the similarity between the left side of (8) and the right side of (2).

Proposition 3. *If $n \geq 1$, and x is a proper coloring of length n, then*

$$\sum_{i=1}^{n} D(n-2i+1) P(\widehat{x}_i) = 0; \tag{8}$$

$$Q(x) = Q^*(x) = P(x) C(n+1). \tag{9}$$

Proof. For the first statement, let R be the set of proper colorings, and $\widehat{x}_A$ be obtained by deleting the entries x_i for $i \in A$ from x. The proof of (8) is by induction on n, the length of x. The identity is easily seen to be true if $n \leq 2$. Suppose that (8) is true for all x of length $n-1$, and let $x \in R$ have length n. For those i with $\widehat{x}_i \in R$, applying (8) gives

$$\sum_{j=1}^{i-1} D(n-2j) P(\widehat{x}_{i,j}) + \sum_{j=i+1}^{n} D(n-2j+2) P(\widehat{x}_{i,j}) = 0. \tag{10}$$

On the other hand, if $\widehat{x}_i \notin R$, then $1 < i < n$ and

$$P(\widehat{x}_{i,j}) = 0 \text{ if } |j-i| > 1 \text{ and } P(\widehat{x}_{i-1,i}) = P(\widehat{x}_{i,i+1}). \tag{11}$$

The left side of (8) for x can be written, using the definition of $P(\widehat{x}_i)$ and then (6), as

$$\begin{aligned}
&= \frac{1}{D(n)} \sum_{\substack{1\le i\le n:\\ \widehat{x}_i\in R}} D(n-2i+1)\Big[\sum_{1\le j<i} C(n-2j)P(\widehat{x}_{i,j}) + \sum_{i<j\le n} C(n-2j+2)P(\widehat{x}_{i,j})\Big] \\
&= \frac{1}{2D(n)} \sum_{\substack{1\le j<i\le n:\\ \widehat{x}_i\in R}} [D(2n-2i-2j+1)+D(2j-2i+1)]P(\widehat{x}_{i,j}) \\
&\quad + \frac{1}{2D(n)} \sum_{\substack{1\le i<j\le n:\\ \widehat{x}_i\in R}} [D(2n-2i-2j+3)+D(2j-2i-1)]P(\widehat{x}_{i,j}).
\end{aligned} \tag{12}$$

Rearranging, and ignoring the $2D(n)$ in the denominator, gives

$$\begin{aligned}
\sum_{i=1}^{n} \mathbf{1}[\widehat{x}_i\in R]\Big[&\sum_{j=1}^{i-1}[D(2n-2i-2j+1)+D(2j-2i+1)]P(\widehat{x}_{i,j}) \\
&+ \sum_{j=i+1}^{n}[D(2n-2i-2j+3)+D(2j-2i-1)]P(\widehat{x}_{i,j})\Big].
\end{aligned} \tag{13}$$

We must show that (10) and (11) imply that (13) is zero.

We would like to write (13) as a linear combination of expressions that vanish because of (10) and (11) as follows.

$$\sum_{\substack{1\le i\le n:\\ \widehat{x}_i\in R}} \alpha_i\Big[\sum_{j=1}^{i-1} D(n-2j)P(\widehat{x}_{i,j}) + \sum_{j=i+1}^{n} D(n-2j+2)P(\widehat{x}_{i,j})\Big] + \sum_{\substack{1\le i\le n:\\ \widehat{x}_i\in R}} \sum_{j=1}^{n} \beta_{i,j}P(\widehat{x}_{i,j}), \tag{14}$$

where $\beta_{i,i} = \beta_{i,i-1} + \beta_{i,i+1} = 0$. If $1 \le i < j \le n$, the coefficient of $P(\widehat{x}_{i,j})$ in (13) is

$$\begin{aligned}
&\mathbf{1}[\widehat{x}_j\in R]\,\big[D(2n-2i-2j+1)+D(2i-2j+1)\big] \\
&+\mathbf{1}[\widehat{x}_i\in R]\,\big[D(2n-2i-2j+3)+D(2j-2i-1)\big].
\end{aligned} \tag{15}$$

The coefficient of $P(\widehat{x}_{i,j})$ in (14) is

$$\mathbf{1}[\widehat{x}_j\in R]\alpha_j D(n-2i) + \mathbf{1}[\widehat{x}_i\in R]\alpha_i D(n-2j+2) + \mathbf{1}[\widehat{x}_i\notin R]\beta_{i,j} + \mathbf{1}[\widehat{x}_j\notin R]\beta_{j,i}. \tag{16}$$

We need to choose the α's and β's so that (15) and (16) agree. If $\widehat{x}_i, \widehat{x}_j \in R$, this says

$$D(2n-2i-2j+1) + D(2n-2i-2j+3) = \alpha_j D(n-2i) + \alpha_i D(n-2j+2)$$

since D is an odd function. It may sound unreasonable to expect to solve this system, since there are n unknowns and $\binom{n}{2}$ equations. However, D satisfies relations that make this possible. Solving the equations for small n suggests trying $\alpha_i = 2C(n-2i+1)$. The fact that this choice solves these equations for all choices of n, i, j then follows from (6) and the fact that D is odd. If $\widehat{x}_i \notin R$ and $\widehat{x}_j \notin R$, (15) and (16) agree if $\beta_{i,j} + \beta_{j,i} = 0$. If $\widehat{x}_i \in R$ and $\widehat{x}_j \notin R$, they agree if

$$D(2n - 2i - 2j + 3) + D(2j - 2i - 1) = \alpha_i D(n - 2j + 2) + \beta_{j,i}.$$

Using (6) again gives $\beta_{j,i} = 2D(2j - 2i - 1)$. Similarly, if $\widehat{x}_i \notin R$ and $\widehat{x}_j \in R$, they agree if $\beta_{i,j} = 2D(2i - 2j + 1)$. With these choices, β is anti-symmetric, and $\beta_{k,k-1} = 2D(1)$ and $\beta_{k,k+1} = 2D(-1)$, so $\beta_{k,k-1} + \beta_{k,k+1} = 0$ as required. This completes the induction argument.

For (9), consider the case of Q first. Use the definition of P to write the right side of (9) as

$$\frac{C(n+1)}{D(n+1)} \sum_{i=1}^{n} C(n - 2i + 1) P(\widehat{x}_i).$$

Using (4), this becomes

$$\frac{1}{2D(n+1)} \sum_{i=1}^{n} C(2n - 2i + 2) P(\widehat{x}_i) + \frac{1}{2} Q(x).$$

Therefore, we need to prove that

$$\sum_{i=1}^{n} \big[C(2n - 2i + 2) - C(2i)\big]\, P(\widehat{x}_i) = 0.$$

But by (5), this follows from (8). The proof for Q^* is similar. □

4. Proof of the main result

We will often write $x_1 x_2 \cdots x_n$ instead of $(x_1, x_2, \ldots, x_n)$ below. If $x \in [q]^m$ and $y \in [q]^n$, let xy denote the word $x_1 \cdots x_m y_1 \cdots y_n \in [q]^{m+n}$

Proof of Theorem 1. We first need to show that the finite dimensional distributions defined in (2) are consistent, i.e., that

$$\sum_{a \in [q]} P(xa) = P(x), \qquad x \in [q]^n,\ n \geq 0. \tag{17}$$

This is true if x is not proper, since then xa is also not proper, and so both sides vanish. For proper x, the proof is by induction on n. Note that for $a \in [q]$,

$$P(a) = \frac{C(0)}{D(2)} = \frac{1}{q},$$

so $\sum_{a \in [q]} P(a) = 1$. This gives (17) for $n = 0$. Suppose it holds for all $x \in [q]^{n-1}$ with $n \geq 1$. Then for proper $x \in [q]^n$, using the induction hypothesis in the second equality,

$$\begin{aligned}\sum_{a \in [q]} P(xa) &= \sum_{a \neq x_n} \frac{1}{D(n+2)} \left[\sum_{i=1}^{n} C(n - 2i + 2) P(\widehat{x}_i a) + C(-n) P(x) \right] \\ &= \frac{1}{D(n+2)} \left[\sum_{i=1}^{n} C(n - 2i + 2) P(\widehat{x}_i) - C(-n + 2) P(x) + (q-1) C(-n) P(x) \right].\end{aligned}$$

The middle term in the second line accounts for the missing term $a = x_n$ when the inductive hypothesis is applied to the case $i = n$ (since $\widehat{x}_n x_n = x$). Using $(j, k, \ell) = (1, n-2i+1, 2i)$ in (7) gives

$$\frac{C(n-2i+2)}{D(n+2)} = \frac{C(n-2i+1)}{D(n+1)} - \frac{C(2i)D(1)}{D(n+2)D(n+1)}.$$

Therefore

$$\sum_{a\in[q]} P(xa) = P(x) - \frac{Q(x)}{D(n+2)} - \frac{C(n-2)}{D(n+2)}P(x) + (q-1)\frac{C(n)}{D(n+2)}P(x).$$

This is $P(x)$, as required, by (9) and the fact that

$$(q-1)C(n) = C(n-2) + C(n+1)D(1),$$

which is obtained by taking $(j, k, \ell) = (2, -n, n+1)$ in (7), and then canceling a factor of $\sqrt{q}$.

Invariance of the measure under permutations of colors and translations is immediate from the definition. Invariance under reflection amounts to checking $P(x) = P(x_n \cdots x_1)$, which follows from the fact that the coefficients of $\widehat{x}_i$ and $\widehat{x}_{n-i+1}$ in (2), which are $C(n-2i+1)$ and $C(-n+2i-1)$ respectively, are equal by the symmetry of C.

For $1-$dependence, we need to show that for $x \in [q]^m$ and $y \in [q]^n$ with $m, n \geq 0$,

$$P(x * y) = P(x)P(y),$$

where the * means that there is no constraint at the single site between x and y. This is again true if x or y is not proper since then both sides are zero. For proper x and y, the proof is by induction, but now on $m + n$. The statement is immediate if $m = 0$ or $n = 0$. So, we take $m \geq 1$ and $n \geq 1$.

There are two cases, according to whether or not xy is a proper coloring, i.e., whether x_m and y_1 are equal or different. Assume first that $x_m = y_1$. Without loss of generality, take their common value to be 1. Then using the definition of P, including the fact that $P(xy) = 0$,

$$\begin{aligned}(18)\quad P(x*y) = \sum_{a\in[q]} P(xay) &= \frac{1}{D(n+m+2)} \sum_{a\neq 1} \Bigg[\sum_{i=1}^{m} C(n+m-2i+2)P(\widehat{x}_i ay) \\ &\qquad + C(n-m)P(xy) + \sum_{j=1}^{n} C(n-m-2j)P(xa\widehat{y}_j)\Bigg] \\ &= \frac{1}{D(n+m+2)}\Bigg[\sum_{i=1}^{m} C(n+m-2i+2)P(\widehat{x}_i * y) + \sum_{j=1}^{n} C(n-m-2j)P(x*\widehat{y}_j)\Bigg].\end{aligned}$$

Using the induction hypothesis, this becomes

$$P(x*y) = \frac{1}{D(n+m+2)}\Bigg[P(y)\sum_{i=1}^{m} C(n+m-2i+2)P(\widehat{x}_i) + P(x)\sum_{j=1}^{n} C(n-m-2j)P(\widehat{y}_j)\Bigg].$$

Taking $(j,k,l)=(n,m-2i+1,i)$ in (7) gives

$$\frac{C(n+m-2i+2)}{D(n+m+2)}=\frac{C(m-2i+1)}{D(m+1)}-\frac{C(2i)D(n+1)}{D(m+1)D(n+m+2)}.$$

Similarly,

$$\frac{C(m+2j-n)}{D(n+m+2)}=\frac{C(2j-n-1)}{D(n+1)}-\frac{C(2n-2j+2)D(m+1)}{D(n+1)D(n+m+2)}.$$

Therefore, since $C(\cdot)$ is even,

$$P(x*y)=P(y)\left[P(x)-\frac{D(n+1)}{D(n+m+2)}Q(x)\right]+P(x)\left[P(y)-\frac{D(m+1)}{D(n+m+2)}Q^*(y)\right].$$

By (9),

$$P(x*y)=P(x)P(y)\left[2-\frac{C(m+1)D(n+1)+C(n+1)D(m+1)}{D(n+m+2)}\right].$$

Taking $(j,k,l)=(n,m-2i+1,i)$ in (7), we see that the expression in brackets above is 1, as required.

Assume now that $x_m\neq y_1$, say $x_m=1$ and $y_1=2$. Then

$$(19)\quad P(x*y)=\sum_{a\in[q]}P(xay)=\frac{1}{D(n+m+2)}\sum_{a\neq 1,2}\left[\sum_{i=1}^{m}C(n+m-2i+2)P(\widehat{x}_iay)\right.$$
$$\left.+C(n-m)P(xy)+\sum_{j=1}^{n}C(n-m-2j)P(xa\widehat{y}_j)\right]$$
$$\frac{1}{D(n+m+2)}\left[\sum_{i=1}^{m}C(n+m-2i+2)P(\widehat{x}_i*y)+\sum_{j=1}^{n}C(n-m-2j)P(x*\widehat{y}_j)\right]$$

as in the previous case. However, in the previous case, the term $P(xy)$ dropped out because xy was not a proper coloring. In this case, the term $(q-2)C(n-m)P(xy)$ is cancelled by the terms $-P(xy)C(n-m+2)$ and $-P(xy)C(n-m-2)$, which arise from

$$\sum_{a\neq 1,2}P(\widehat{x}_may)=P(\widehat{x}_m*y)-P(xy)\text{ and }\sum_{a\neq 1,2}P(xa\widehat{y}_1)=P(x*\widehat{y}_1)-P(xy).$$

The fact that the overall coefficient of $P(xy)$ vanishes is a consequence (4) with $m=2$, since $2C(2)=q-2$. The rest of the proof is the same as in the case $x_m=y_1$ above. □

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